A TEST-RATIO TEST FOR CONTINUED FRACTIONS*

WALTER LEIGHTON

Introduction. The general question of convergence of continued fractions of the form $1+K_1^{\infty}[b_n/1]$ remains in a large measure unanswered, even though continued fractions of this type are of especial importance from a function-theoretic point of view. Valuable contributions have been made by E. B. Van Vleck, A. Pringsheim, O. Szász, O. Perron, and others. Leighton and Wall [7] recently gave new types of convergence criteria for continued fractions of this kind. Jordan and Leighton in a paper to be published soon give a large number of new sets of sufficient conditions for convergence.

The purpose of the present paper is to establish the first test-ratio test for continued fractions and a very general theorem on convergence, which is also believed to be the first of its kind. This test leads to a class of continued fractions, the *precise* region of convergence of which is the interior of a circle. This is a new phenomenon.

1. **A test-ratio test.** Let

$$1+\overset{\infty}{\underset{1}{K}}[b_n/1]=1+\frac{b_1}{1}+\frac{b_2}{1}+\cdots \tag{1.1}$$

be a continued fraction in which the b_n are complex numbers $\neq 0$.

THEOREM 1. *If the ratio $|b_{n+1}/b_n|$ is less than or equal to $k<1$ for n sufficiently large, the continued fraction* (1.1) *converges at least in the wider sense. If $|b_{n+1}/b_n|$ is greater than or equal to $1/k>1$ for n sufficiently large, the continued fraction diverges by oscillation. If the limit of the ratio is unity, the continued fraction may converge or diverge.*

Suppose $|b_{n+1}/b_n|\leqq k<1$ for n sufficiently large. It follows that there exists a positive integer N such that $|b_n|<1/4$ for $n\geqq N$. Each continued fraction $K_n^{\infty}[b_n/1]$ then converges (Van Vleck [2], Pringsheim [4]) for $n\geqq N$. The proof of the first statement of the theorem is complete.

Assume $|b_{n+1}/b_n|\geqq 1/k>1$ for n sufficiently large. Write (1.1) in the equivalent form (Perron [8], p. 197)

$$1+\overset{\infty}{\underset{1}{K}}[1/a_n], \tag{1.2}$$

* Presented to the Society, September 1, 1936.

where

$$a_{2n} = \frac{b_1 b_3 \cdots b_{2n-1}}{b_2 b_4 \cdots b_{2n}}, \quad a_{2n-1} = \frac{b_2 b_4 \cdots b_{2n-2}}{b_1 b_3 \cdots b_{2n-1}}, \quad n = 1, 2, 3, \cdots; b_0 \equiv 1.$$

It will be shown that the series $\sum |a_n|$ converges, and it will follow from a theorem of Stern [1] that the continued fraction (1.2), and hence (1.1), diverges by oscillation. It is sufficient to observe that

$$|a_{2n}/a_{2n-2}| = |b_{2n-1}/b_{2n}| \leqq k < 1,$$
$$|a_{2n+1}/a_{2n-1}| = |b_{2n}/b_{2n+1}| \leqq k < 1,$$

for n sufficiently large. Thus the two series $\sum |a_{2n+1}|$ and $\sum |a_{2n}|$ converge. It follows that the series $\sum |a_n|$ converges, and the second statement of the theorem follows as indicated.

To prove the final statement of the theorem, it is sufficient to consider the example

$$1 + \frac{a}{1} + \frac{a}{1} + \cdots. \tag{1.3}$$

When $a=1$ it is well known that this continued fraction converges to the value $(1+5^{1/2})/2$. When $a=-1$, a computation of the successive approximants proves immediately that the continued fraction diverges. Indeed, Szász [6] has shown that the continued fraction (1.3) diverges for every $\epsilon>0$, if $a=-\epsilon-1/4$.

COROLLARY. *If* $\lim_{n\to\infty}|b_{n+1}/b_n| = k$, *the continued fraction* (1.1) *will converge, at least in the wider sense, if* $k<1$, *and will diverge if* $k>1$.

The proof is immediate.

EXAMPLE. *A continued fraction with a circle as its region of convergence.* Consider the continued fraction

$$1 + \mathop{K}_{1}^{\infty}[c_n x^n/1] = 1 + \frac{c_1 x}{1} + \frac{c_2 x^2}{1} + \cdots, \tag{1.4}$$

where the c_n are complex nonzero numbers. If $\lim_{n\to\infty}|c_{n+1}/c_n| = c \neq 0$, it follows from the preceding corollary that the continued fraction (1.4) converges, at least in the wider sense, to a function analytic except possibly for a finite number of poles in every closed region wholly interior to the circle $|x| = 1/c$, and diverges outside. Further, if $\lim_{n\to\infty}|c_{n+1}/c_n| = 0$, (1.4) converges to a function meromorphic throughout the finite plane.

2. **A general theorem on convergence.** Leighton and Wall [7] gave an example of a convergent continued fraction (1.1) where the elements b_n were everywhere dense in the complex plane. The following theorem attacks the general question of convergence from a different point of view. We assume as usual that all $b_n \neq 0$.

THEOREM 2. *Let $m_0, m_1, m_2, \cdots$ be any sequence of positive integers such that $m_0 = 2$, $m_{n+1} - m_n \geqq 2$, $(n = 0, 1, 2, \cdots)$. The numbers*

$$b_{m_0}, b_{m_1}, b_{m_2}, \cdots \tag{2.1}$$

can be chosen in such a fashion that with at most one value in the complex plane excluded from each of the numbers b_n not contained in the set (2.1), *the continued fraction* (1.1) *will converge.*

Let A_n/B_n represent the nth approximant of (1.1), where A_n and B_n are given by the usual recursion relations

$$\begin{aligned} &A_0 = 1, \quad B_0 = 1, \quad A_1 = 1 + b_1, \quad B_1 = 1, \\ &A_n = A_{n-1} + b_n A_{n-2}, \\ &B_n = B_{n-1} + b_n B_{n-2}, \qquad n = 2, 3, 4, \cdots. \end{aligned} \tag{2.2}$$

By means of (2.2) write A_j and B_j, $(j = 2, 3, \cdots, m_1 - 1)$, as

$$A_j = f_0^j A_1 + b_2 g_0^j A_0, \qquad B_j = f_0^j B_1 + b_2 g_0^j B_0, \tag{2.2'}$$

where f_0^j and g_0^j are polynomials in the numbers $b_3, b_4, \cdots, b_{m_1-1}$ and do not depend on any other b's. (Perron [8], p. 14, uses the symbol A_{t-m_r,m_r} for f_r^t, and B_{t-m_r,m_r} for g_r^t). Suppose the numbers f_0^j are nonzero. It is clear that $|b_{m_0}| = |b_2|$ can be chosen so small that simultaneously

$$\left| \frac{A_j}{B_j} - \frac{A_1}{B_1} \right| < \frac{1}{2}, \quad j = 2, 3, \cdots, m_1 - 1.$$

Now write A_k and B_k, $(k = m_1, m_1 + 1, \cdots, m_2 - 1)$, as

$$\begin{aligned} A_k &= f_1^k A_{m_1-1} + b_{m_1} g_1^k A_{m_1-2}, \\ B_k &= f_1^k B_{m_1-1} + b_{m_1} g_1^k B_{m_1-2}, \end{aligned} \tag{2.2''}$$

where f_1^k and g_1^k are polynomials in $b_{m_1+1}, b_{m_1+2}, \cdots, b_{m_2-1}$ and do not depend on any b's not in this set. Similarly, let us suppose for the moment that the numbers f_1^k are never zero. The number $|b_{m_1}|$ can then be taken so small that

$$\left| \frac{A_k}{B_k} - \frac{A_{m_1-1}}{B_{m_1-1}} \right| < \frac{1}{2^2}, \qquad k = m_1, m_1 + 1, \cdots, m_2 - 1.$$

Continue the process. With the assumption that f_r^t is never zero it is clear that $|b_{m_r}|$ can be chosen so small that

$$\left| \frac{A_t}{B_t} - \frac{A_{m_r-1}}{B_{m_r-1}} \right| < \frac{1}{2^{r+1}}, \qquad t = m_r, m_r + 1, \cdots, m_{r+1} - 1.$$

The continued fraction will thus converge.

It remains to assign conditions to the numbers b_n so that the numbers f_r^t will be different from zero. It is sufficient to exclude precisely one value in the finite complex plane from each b_n not in the set $b_{m_0}, b_{m_1}, \cdots$. For, in the general case, it follows from (2.2) that

$$(2.3) \qquad \begin{aligned} &f_r^{m_r} = 1, \qquad f_r^{m_r+1} = 1 + b_{m_r+2}, \\ &f_r^{m_r+s} = f_r^{m_r+s-1} + b_{m_r+s} f_r^{m_r+s-2}, \quad s = 2, 3, \cdots, m_{r+1} - m_r - 1, \end{aligned}$$

where $f_r^{m_r+s-1}$ is a polynomial in $b_{m_r+2}, b_{m_r+3}, \cdots, b_{m_r+s-1}$ and depends on no other b's. The value -1 is first excluded from b_{m_r+2}. It follows from (2.3) that one value may be excluded from each successive b in such a way that f_r^t is never zero. This completes the proof of the theorem.

Bibliography

1. M. A. Stern, *Lehrbuch der algebraischen Analysis*, Leipzig, 1860.
2. E. B. Van Vleck, *On the convergence and character of the continued fraction* $K_1^{\infty}[a_n z/1]$. Transactions of this Society, vol. 2 (1901), pp. 215–233.
3. A. Pringsheim, *Über die Konvergenz unendlicher Kettenbrüche*, Sitzungsberichte der Akademie der Wissenschaften zu München, vol. 28 (1898).
4. ———, *Über einige Konvergenzkriterien für Kettenbrüche mit komplexen Gliedern*, ibid., vol. 35 (1905).
5. O. Szász, *Über gewisse unendliche Kettenbruch-Determinanten und Kettenbrüche mit komplexen Elementen*, ibid., vol. 42 (1912).
6. ———, *Über die Erhaltung der Konvergenz unendlicher Kettenbrüche bei independenter Veränderlichkeit aller ihrer Elemente*, Journal für die reine und angewandte Mathematik, vol. 147 (1916), pp. 132–160.
7. W. Leighton and H. S. Wall, *On the transformation and convergence of continued fractions*, American Journal of Mathematics, vol. 58 (1936).
8. O. Perron, *Die Lehre von den Kettenbrüchen*, Leipzig, 1929.

The Rice Institute

CERTAIN INCOMPLETE NUMERICAL FUNCTIONS*

W. A. DWYER

1. Introduction. In a recent paper,† the author has shown that a certain pair of identities, given by Uspensky,‡ involving incomplete numerical functions in three variables result from a more general formula, which is obtained by the application of the method of paraphrase§ to a certain theta function identity. Uspensky's arithmetic proofs of his formulas are simple but give no suggestion of a systematic procedure for determining other identities of the same type. The analytic proof, however, provides a method by which related identities can be obtained by comparatively simple and straightforward operations.

The theta function identity (identity I, §2) mentioned above involves a product of the form

$$\theta_\alpha(x + y + z)\phi_{\alpha\beta\gamma}(u, v),$$

where

$$\phi_{\alpha\beta\gamma}(u, v) = \theta_1' \frac{\theta_\alpha(u + v)}{\theta_\beta(u)\theta_\gamma(v)}$$

(α, β, γ are restricted to certain triads of the numbers 0, 1, 2, 3) is one of the sixteen doubly periodic functions of the second kind.‖ Sixty-four products of this type are possible, and each leads to a theta identity which, when paraphrased, yields a formula of the same general type as Uspensky's. Any one of these theta identities may be derived from any other one by elementary transformations (of the form $x = x + \pi/2 + \pi\tau/2$, for example) on one or more of the arguments. However, the transformations involving $\pi\tau/2$ are awkward. This suggests a basic set of identities from which all others may be obtained by comparatively easy transformations involving $\pi/2$ only.

---

* Much of the material appearing in this paper is summarized in an abstract entitled *On certain pseudo-periodic functions*, presented to the Society, April 10, 1937.

† American Journal of Mathematics, vol. 59 (1937), pp. 290–294.

‡ J. V. Uspensky, *Sur les relations entre les nombres des classes des formes quadratiques binaires et positives*, Bulletin de l'Académie des Science de l'URSS, 1925–1926. Also, American Journal of Mathematics, vol. 50 (1928), pp. 93–122; this Bulletin, vol. 36 (1930), pp. 743–754. The formulas mentioned above are labeled (III) and (IV) in this last paper.

§ E. T. Bell, Transactions of this Society, vol. 22 (1921), pp. 1–30; 198–219.

‖ Hermite's nomenclature, *Oeuvres*, vol. 4, pp. 199–200.

In this paper, we shall concern ourselves with this basic set of eight related theta identities and their paraphrases.

2. **The fundamental theta identities.** The procedure* used in establishing the theta identities appears in the proof of our identity I in the first paper; so only the results will be given here. We shall introduce the notation

$$\chi(x, y) = \sum_{r=-\infty}^{\infty} q^{r^2} e^{-2iry} \cot (x - r\pi\tau),$$

$$\chi_1(x, y) = \sum_{m=-\infty}^{\infty} q^{m^2/4} e^{-miy} \cot (x - m\pi\tau/2),$$

$$\chi_2(x, y) = \sum_{r=-\infty}^{\infty} q^{r^2} e^{-2iry} \csc (x - r\pi\tau),$$

$$\chi_3(x, y) = \sum_{m=-\infty}^{\infty} q^{m^2/4} e^{-miy} \csc (x - m\pi\tau/2),$$

where r is arbitrary and integral and m is odd.

Then the eight fundamental identities are as follows:

I. $\theta_3(x+y+z)\phi_{111}(x+y, -y) = \theta_3(z)\chi(x+y, x+z) - \theta_3(x+z)\chi(y, z).$

II. $\theta_2(x+y+z)\phi_{111}(x+y, -y) = \theta_2(z)\chi_2(x+y, x+z) - \theta_2(x+z)\chi_2(y, z).$

III. $\theta_3(x+y+z)\phi_{100}(x+y, -y) = \theta_2(z)\chi_1(x+y, x+z) - \theta_2(x+z)\chi_1(y, z).$

IV. $\theta_2(x+y+z)\phi_{100}(x+y, -y) = \theta_3(z)\chi_3(x+y, x+z) - \theta_3(x+z)\chi_3(y, z).$

V. $\theta_2(x+y+z)\phi_{001}(x+y, -y) = \theta_3(z)\chi_1(x+y, x+z) - \theta_2(x+z)\chi(y, z).$

VI. $\theta_3(x+y+z)\phi_{001}(x+y, -y) = \theta_2(z)\chi_3(x+y, x+z) - \theta_3(x+z)\chi_2(y, z).$

VII. $\theta_2(x+y+z)\phi_{010}(x+y, -y) = \theta_2(z)\chi(x+y, x+z) - \theta_3(x+z)\chi_1(y, z).$

VIII. $\theta_3(x+y+z)\phi_{010}(x+y, -y) = \theta_3(z)\chi_2(x+y, x+z) - \theta_2(x+z)\chi_3(y, z).$

The fifty-six remaining identities of this type may be obtained by means of elementary transformations, of the form $x \to x+\pi/2$, applied to one or more of the variables.

For completeness we shall list the sixty-four expressions and the transformations by which any one may be obtained from some one of the given identities. Let the transformations representing an increase of $\pi/2$ in the respective variables be designated as follows:

* The method is essentially that given by Appell for his theory of the doubly periodic functions of the third kind, as has been shown by M. A. Basoco, Acta Mathematica, vol. 57, pp. 95–100. Cf. also, M. A. Basoco, American Journal of Mathematics, vol. 54 (1932), pp. 242–252.

(a) z; (b) x; (c) y; (d) x and z; (e) y and z; (f) x and y; (g) x, y, and z. By means of this notation the results may be given in a table:

$\phi_{\alpha\beta\gamma}$	θ_1		θ_2	θ_3		θ_0
111	a		0	0		a
212	g		f	f		g
221	b	II	d	d	I	b
122	c		e	e		c
100	a		0	0		a
203	g		f	f		g
230	b	IV	d	d	III	b
133	c		e	e		c
001	a		0	0		a
302	g		f	f		g
331	b	V	d	d	VI	b
032	c		e	e		c
010	a		0	0		a
313	g		f	f		g
320	b	VII	d	d	VIII	b
023	c		e	e		c

Thus, for example, $\theta_1(x+y+z)\theta_{132}(x+y, -y)$ can be obtained by applying transformation (c) (that is, letting y be increased by $\pi/2$) to identity IV.

3. **The paraphrases of the eight identities.** The formulas and the procedure made use of in the paraphrase of I are set forth in the earlier paper, and since the arithmetized expressions needed in the paraphrase of the remaining identities are available in the literature,* the final results only will be given.

* For arithmetized forms of the theta and phi functions, see, for example, E. T. Bell, Messenger of Mathematics, vol. 54 (1924). The arithmetized forms of cot $(x-n\pi\tau)$, and so on, may be obtained from the exponential forms but are given directly in a paper by M. A. Basoco, American Journal of Mathematics, vol. 54 (1932). The trigonometric reduction formulas may be found in E. T. Bell's *Arithmetical paraphrases*, Transactions of this Society, vol. 22 (1921), pp. 198–219, or built up from formulas appearing in this paper; see also Bell's paper in the Giornale di Matematiche, vol. 61 (1923).

With regard to the notation, the finite summations appearing in each identity refer to all integral solutions of the corresponding partitions listed with that identity, subject to the following conditions: i, h arbitrary; $\mu \gtrless 0$ odd; d, δ, t, τ, m, n, r, s, $k>0$, m, τ odd; and

$$\begin{aligned} &0 < \Delta_0 < \Delta_0', && \Delta_0' - \Delta_0 \equiv 0 \pmod 2,\\ &0 < \Delta_1 < \Delta_1', && \Delta_1' - \Delta_1 \equiv 1 \pmod 2,\\ &0 < \tau_0 < t_0, && t_0 - \tau_0 \equiv 0 \pmod 4,\\ &0 < \tau_2 < t_2, && t_2 - \tau_2 \equiv 2 \pmod 4. \end{aligned}$$

Also, $e(n)=1$ or 0 according as n is or is not a perfect square, and $a(n)=1$ or 0 according as n is or is not the sum of two squares.

Several of the identities given below, together with their partitions, could quite obviously be simplified, but it is believed best to present them in the form resulting directly from the paraphrase.*

In all cases $F(x, y, z)$ is a function defined for integral values of the arguments and subject to the parity conditions;

$$F(-x, -y, -z) = -F(x, y, z), \qquad F(0, 0, 0) = 0.$$

The identities are as follows:

$$\begin{aligned} &\sum F(\delta+i, \delta-d+i, i) = \sum \{F((\Delta_0'+\Delta_0)/2, (\Delta_0'-\Delta_0)/2, \Delta_0-h)\\ &\quad -F(-h, (\Delta_0'-\Delta_0)/2, \Delta_0-h)\}\\ \text{I}' \qquad &\quad +e(n)\sum_{j=1}^{s-1}\{F(j, j, s)-F(s, j, s)\}+a(n)\sum F(r, 0, r-k),\\ &N = i^2+2d\delta = h^2+\Delta_0\Delta_0' = s^2 = r^2+k^2. \end{aligned}$$

$$\begin{aligned} &\sum F(\mu+2d, \mu+2d-2\delta, \mu) = \sum \{F(\Delta_1'+\Delta_1, \Delta_1'-\Delta_1, \mu+2\Delta_1)\\ &\quad -F(\mu, \Delta_1'-\Delta_1, \mu+2\Delta_1)\}\\ \text{II}' \qquad &\quad +e(n)\sum_{j=1}^{(m-1)/2}\{F(2j-1, 2j-1, m)-F(m, 2j-1, m)\},\\ &N = \mu^2+8d\delta = \mu^2+4\Delta_1\Delta_1' = m^2. \end{aligned}$$

* The paraphrased identities can be derived from each other by simpler considerations. For example, to derive II′ from I′ we may let $F(x, y, z)=0$ unless x, y, z are odd; and like manipulations allow other identities to be derived from any one. Such relationships are not unexpected, since these identities are the paraphrases of interdependent theta identities. However, in the interest of simplicity of presentation and of possible applications it is believed better to present them in the form in which they appear.

$$\text{III}'\qquad \begin{aligned}&\sum F(t+2i, t-\tau+2i, 2i)=\sum\{F((t_0+\tau_0)/2, (t_0-\tau_0)/2, \mu+\tau_0)\\&\qquad -F(\mu, (t_0-\tau_0)/2, \mu+\tau_0)\}+a(n)\sum F(m_1, 0, m_1-m_2),\\&N=4i^2+2t\tau=\mu^2+t_0\tau_0=m_1^2+m_2^2,\ t, \tau \text{ odd}.\end{aligned}$$

$$\text{IV}'\qquad \begin{aligned}&\sum F(\mu+t, \mu+t-\tau, \mu)=\sum\{F((t_2+\tau_2)/2, (t_2-\tau_2)/2, 2i+\tau_2)\\&\qquad -F(2i, (t_2-\tau_2)/2, 2i+\tau_2)\},\\&N=\mu^2+2t\tau=4i^2+t_2\tau_2,\ t, \tau \text{ odd}.\end{aligned}$$

$$\text{V}'\qquad \begin{aligned}&\sum F(2t+\mu, 2t+\mu-\tau, \mu)=\sum F((t_0+\tau_0)/2, (t_0-\tau_0)/2, 2i+\tau_0)\\&\qquad -\sum F(\mu, \Delta_0'-\Delta_0, \mu+2\Delta_0)\\&\qquad -e(n)\sum_{j=1}^{(m-1)/2} F(m, 2j, m)+a(n)\sum F(m, 0, m-2r),\\&N=\mu^2+4t\tau=4i^2+t_0\tau_0=\mu^2+4\Delta_0\Delta_0'=m^2=m^2+4r^2.\end{aligned}$$

$$\text{VI}'\qquad \begin{aligned}&\sum F(2i+2t, 2i+2t-\tau, 2i)=\sum F((t_2+\tau_2)/2, (t_2-\tau_2)/2, \mu+\tau_2)\\&\qquad -\sum F(2i, \Delta_1'-\Delta_1, 2i+2\Delta_1)-e(n)\sum_{j=1}^{s} F(2s, 2j-1, 2s),\\&N=i^2+t\tau=\frac{\mu^2+t_2\tau_2}{4}=i^2+\Delta_1\Delta_1'=s^2.\end{aligned}$$

$$\text{VII}'\qquad \begin{aligned}&\sum F(\mu+\tau, \mu-2t+\tau, \mu)=\sum F(\Delta_0'+\Delta_0, \Delta_0'-\Delta_0, \mu+2\Delta_0)\\&\qquad -\sum F(2i, (t_0-\tau_0)/2, 2i+\tau_0)\\&\qquad +e(n)\sum_{j=1}^{(m-1)/2} F(2j, 2j, m)+a(n)\sum F(2r, 0, 2r-m),\\&N=\mu^2+4t\tau=\mu^2+4\Delta_0\Delta_0'=4i^2+t_0\tau_0=m^2=m^2+4r^2.\end{aligned}$$

$$\text{VIII}'\qquad \begin{aligned}&\sum F(2i+\tau, 2i-2t+\tau, 2i)=\sum F(\Delta_1'+\Delta_1, \Delta_1'-\Delta_1, 2h+2\Delta_1)\\&\qquad -\sum F(\mu, (t_2-\tau_2)/2, \mu+\tau_2)+e(n)\sum_{j=1}^{s} F(2j-1, 2j-1, 2s),\\&N=i^2+t\tau=h^2+\Delta_1\Delta_1'=\frac{\mu^2+t_2\tau_2}{4}=s^2.\end{aligned}$$

If we restrict the parity conditions in I′ by means of the relations

$F(-x, y, z)=F(x, y, z)$, $F(x, -y, -z)=-F(x, y, z)$, $F(x, 0, z)=0$,

and

$F(-x, y, z)=-F(x, y, z)$, $F(x, -y, -z)=F(x, y, z)$, $F(x, 0, z)=0$,

we obtain the two identities of Uspensky mentioned earlier.

4. **The number of representations.** Expressions yielding the number of representations of any number as the sum of three squares in terms of divisor functions can be easily derived from the identities.

We shall make use of the following incomplete numerical functions:

$$g(n) = \sum (-1)^{(d-1)/2}, \qquad n = d\delta,\ 0 < d < \delta,\ d \text{ odd},$$
$$h(n) = \sum (-1)^{(\delta-d-1)/2},\ n = d\delta,\ 0 < d < \delta,\ \delta - d \equiv 1 \pmod 2,$$

and define

$$G(n) = g(n) + 2g(n-2^2) + 2g(n-4^2) + \cdots,$$
$$G'(n) = g(n-1^2) + g(n-3^2) + g(n-5^2) + \cdots,$$
$$H(n) = h(n/4) - 2h((n-4\cdot 1^2)/4) + 2h((n-4\cdot 2^2)/4) - 2h((n-4\cdot 3^2)/4) + \cdots.$$

Then, letting

$$F(x, y, z) = (-1)^{x/2+(z-1)/2}, \quad F(x, y, z) = (-1)^{(x-1)/2+z/2},$$
$$F(x, y, z) = (-1)^{x/2-(y-1)/2},$$

in identities IV′ and VII′, in III′, and in VI′, respectively, and letting $N_3(n\equiv k)$ denote the number of representations of $n\equiv k \pmod 8$ as the sum of three squares, we obtain the following theorems:

THEOREM 1. *From* IV′, $N_3(n\equiv 3)=8G(n)$.

This result is due to Liouville and has been obtained by Uspensky from a special case of identity I′.

THEOREM 2. $N_3(n\equiv 1)=12G(n)-12M+12\sum(-1)^{(m-1)/2}$,* *from* VII′, *where* M *is* $+1$ *if* n *is a square of a number of the form* $4k+3$ *and is zero otherwise, and the last sum is over the positive solutions of the equation* $n=m^2+4r^2$.

THEOREM 3. *Also from* VII′, $N_3(n\equiv 5)=12G(n)+12\sum(-1)^{(m-1)/2}$.

THEOREM 4. *From* III′, $N_3(n\equiv 2)=24G'(n)+24K$, *where* K *is equal to* $+1$ *or* -1 *according as* n *is the sum of the squares of two numbers both*

* The six possible solutions for a square $x^2=x^2+0+0$ are not counted by this formula nor by the starred formulas in Theorems 6 and 7.

of the form $4k+1$ *or both of the form* $4k+3$, *and is zero otherwise.*

THEOREM 5. *From* III', $N_3(n\equiv 6)=24G'(n)$.

THEOREM 6. *From* VI', $N_3(n\equiv 0)=8G'(n)+4H(n)-4e(n)$.*

THEOREM 7. *From* VI', $N_3(n\equiv 4)=8G'(n)+4H(n)-8e(n)$.*

Since $n\equiv 7 \pmod 8$ cannot be represented as the sum of three squares, the set of formulas is complete.

It is clear that by selecting other functions $F(x, y, z)$ in a suitable manner other arithmetical results implicit in our general formulas may be obtained. As is usually the case in results of this type, strictly elementary proofs are no doubt possible, but are sometimes difficult to establish even after the theorems are known.

CREIGHTON UNIVERSITY

INVOLUTORY SYSTEMS OF CURVES ON RULED SURFACES†

A. F. CARPENTER

In a paper presented to the International Mathematical Congress in Toronto in 1924, N. B. McLean discussed the properties of a certain one-parameter family of curves lying upon a ruled surface and characterized by the condition of forming a constant cross ratio with the complex curves of the surface. It is the purpose of the present paper to generalize McLean's system of curves and then to call attention to certain interesting special cases.

For the defining system of differential equations we make use of the form

$$\text{(R)} \qquad y'' + p_{12}z' + q_{11}y + q_{12}z = 0, \quad z'' + p_{21}y' + q_{21}y + q_{22}z = 0,$$

where $p'_{12}=2q_{12}$, $p'_{21}=2q_{21}$. For this form the two directrix curves C_y, C_z are the two branches of the flecnode curve of R.

The tetrahedron of reference is that determined by the four points P_y, P_z, P_ρ, P_σ, where

$$\text{(1)} \qquad \rho = 2y' + p_{12}z, \qquad \sigma = 2z' + p_{21}y,$$

the unit point being so chosen that the general point of space will be represented by the expression

$$x_1y + x_2z + x_3\rho + x_4\sigma.$$

† Presented to the Society, April 16, 1938.

The curves generated by the points $\eta=\alpha y+\beta z$, $\zeta=\alpha y-\beta z$, where α, β are arbitrary functions of the independent variable x, cut each generator g of R in points harmonically separated by the flecnode points of that generator. If the tangents to the four curves C_y, C_z, C_η, C_ζ, at points in which they cut a generator of R are to lie on a quadric, then the matrix of the coordinates of the general points on these tangents must be of rank two. The general points on these four tangent lines are given by the expressions

$$y + 2\kappa y' = y - p_{12}\kappa z + \kappa\rho, \qquad z + 2\lambda z' = -p_{21}\lambda y + z + \lambda\sigma,$$
$$\eta + 2\mu\eta' = (\alpha + 2\mu\alpha' - p_{21}\beta\mu)y + (\beta + 2\mu\beta' - p_{12}\alpha\mu)z + \alpha\mu\rho + \beta\mu\sigma,$$
$$\zeta + 2\nu\zeta' = (\alpha + 2\nu\alpha' + p_{21}\beta\nu)y - (\beta + 2\nu\beta' + p_{12}\alpha\nu)z + \alpha\nu\rho - \beta\nu\sigma,$$

and the rank of the matrix of the coefficients of y, z, ρ, σ in these expressions is two when

$$\mu = \nu = \kappa\alpha/(\alpha - 2\kappa\alpha') = \lambda\beta/(\beta - 2\lambda\beta'). \tag{2}$$

When conditions (2) are fulfilled the equation of the corresponding quadric is found to be

$$x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2 - 2(\alpha'/\alpha - \beta'/\beta)x_3x_4 = 0. \tag{3}$$

If now $\alpha=c_1\theta_1(x)$, $\beta=c_2\theta_2(x)$, where θ_1, θ_2 are arbitrary functions and c_1, c_2 constants, then

$$\alpha'/\alpha - \beta'/\beta = \theta_1'/\theta_1 - \theta_2'/\theta_2,$$

and it follows that *there exists on R, for each choice of the ratio θ_1/θ_2 a one-parameter family of curves forming an involution such that the double elements are the two branches of the flecnode curve, and such that the tangents to the curves of each family at points of intersection with a generator of R constitute one regulus of a quadric which is itself uniquely determined by the choice of θ_1/θ_2.*

If $\theta_1=p_{21}^{1/2}$ and $\theta_2=p_{12}^{1/2}$, quadric (3) becomes

$$p_{12}p_{21}(x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2) + 2(p_{12}q_{21} - p_{21}q_{12})x_3x_4 = 0, \tag{4}$$

the flecnode-complex quadric defined by McLean. The curves C_η, C_ζ are now generated by the points

$$\eta = c_1p_{21}^{1/2}y + c_2p_{12}^{1/2}z, \qquad \zeta = c_1p_{21}^{1/2}y - c_2p_{12}^{1/2}z.$$

When $c_1=1$, $c_2=1$, C_η and C_ζ are the complex curves of R. On the other hand, when $c_1=1$, $c_2=i$, the points determined by η, ζ are the involute points of g, and C_η, C_ζ are the involute curves of R. McLean has shown that the quadric (4) may be determined by the generator g,

the line of intersection of the planes osculating the two branches of the flecnode curve at their points of intersection with g, and the line of intersection of the planes osculating the two branches of the complex curve at their points of intersection with g. In view of the above facts, the theorem is equally true if for the complex curve we substitute the involute curve. Since for every generator on which the flecnode points are distinct either the complex points are real or the involute points are real, but not both pairs, the mystery of the existence of the quadric (4) when the complex points are imaginary is explained.

By definition all the quadrics given by (3) are tangent to R along g. Since, no matter how α and β are chosen, the points P_y, P_z are double points of the involution on g, the tangents to C_y, C_z at these points lie on each quadric. Thus all the quadrics have as a common intersection the generator g of R counted twice, together with the tangents at P_y, P_z to the two branches of R's flecnode curve.

On each of the quadrics given by (3) lie, in general, two asymptotic tangents of R. That regulus of R's osculating quadric

$$x_1x_4 - x_2x_3 = 0 \tag{5}$$

whose lines intersect g consists of asymptotic tangents of R. From (3) and (5) it is seen that the pair of planes

$$p_{12}x_3^2 + 2(\alpha'/\alpha - \beta'/\beta)x_3x_4 - p_{21}x_4^2 = 0 \tag{6}$$

contains the complete intersection of (3) and (5). These planes intersect in g and are tangent planes to (5), their points of tangency being given by the expressions

$$\{\beta'/\beta - \alpha'/\alpha \pm [(\beta'/\beta - \alpha'/\alpha)^2 + p_{12}p_{21}]^{1/2}\}y + p_{12}z. \tag{7}$$

Asymptotic tangents to R at these points thus lie on the quadrics given by (3).

If we suppose in particular that $\alpha'/\alpha - \beta'/\beta = 0$, so that $\alpha/\beta = c_1/c_2 = \text{const.}$, then (7) becomes, after division by $\pm p_{12}^{1/2}$,

$$p_{21}^{1/2}y \pm p_{12}^{1/2}z.$$

But these are the expressions for the complex points of g.* *There is thus seen to exist on R a one-parameter involutory family of curves whose double curves are the two branches of the flecnode curve and which are such that any four curves of the family cut all generators of R in constant cross ratio. The tangents to these curves along any generator con-*

* Wilczynski, *Projective Differential Geometry*, p. 207.

stitute one regulus of a quadric whose equation may be written in the form

$$x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2 = 0, \tag{8}$$

and on this quadric lie the asymptotic tangents of R at the complex points of g. That regulus of this quadric which consists of the tangents to the η, ζ curves is generated by the two axial pencils

$$x_3 - \kappa x_4 = 0, \qquad x_1 - \kappa(x_2 + p_{12}x_3) + p_{21}x_4 = 0,$$

whose axes are, respectively, the generator g and the line joining $P_{y'}$ to $P_{z'}$. The other regulus is generated by the two axial pencils

$$x_2 + p_{12}x_3 - \lambda x_4 = 0, \qquad x_1 - \lambda x_3 + p_{21}x_4 = 0,$$

whose axes are, respectively, the tangents to the two branches of the flecnode curve at P_y, P_z.

Without going into further detail we may state that when $\eta = c_1y + c_2z$, $\zeta = c_1y - c_2z$ the lines of intersection of the osculating planes of pairs of curves C_η, C_ζ, at points P_η, P_ζ constitute one regulus of a quadric whose equation in the reference system here employed, where $p = p_{12}q_{21} - p_{21}q_{12}$, $q = q_{11} - q_{22}$, is

$$p(x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2) - 2q(p_{12}x_1x_3 - p_{21}x_2x_4 - q_{12}x_3^2 - qx_3x_4 + q_{21}x_4^2) = 0.$$

This quadric also is seen to be tangent to R along g and to have in common with (5) and (8) the two asymptotic tangents at the complex points of g.

Let us next assume that $\alpha'/\alpha - \beta'/\beta = c$, and hence that $\alpha/\beta = ae^{cz}$.

Equation (3) now becomes

$$x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2 - 2cx_3x_4 = 0, \tag{9}$$

which appears to define a one-parameter family of quadrics. But a transformation $\xi = cx$ of the independent variable, which has no geometric significance, effects the following transformations in the coefficients of system (R) and in the reference system:

$$p_{ik} = c\bar{p}_{ik}, \quad q_{ik} = c^2\bar{q}_{ik}, \quad x_1 = \bar{x}_1, \quad x_2 = \bar{x}_2, \quad cx_3 = \bar{x}_3, \quad cx_4 = \bar{x}_4.$$

It follows that the constant c drops out of equation (9) and we have

$$x_1x_4 - x_2x_3 - p_{12}x_3^2 + p_{21}x_4^2 - 2x_3x_4 = 0; \tag{10}$$

so no generality is lost by taking $c = 1$.

If we write $u = ae^z$ so that $\eta = u^{1/2}y + u^{-1/2}z$, $\zeta = u^{1/2}y - u^{-1/2}z$, the system of differential equations in η and ζ becomes

$$
\begin{aligned}
&\eta'' + \tfrac{1}{2}(p_{12}u + p_{21}u^{-1})\eta' - \tfrac{1}{2}(p_{12}u - p_{21}u^{-1} + 2)\zeta' \\
&\quad + \tfrac{1}{4}(p_{12}u - p_{21}u^{-1} + 2q_{12}u + 2q_{21}u^{-1} + 2q_{11} + 2q_{22} + 1)\eta \\
&\quad - \tfrac{1}{4}(p_{12}u + p_{21}u^{-1} + 2q_{12}u - 2q_{21}u^{-1} - 2q)\zeta = 0, \\
&\zeta'' + \tfrac{1}{2}(p_{12}u - p_{21}u^{-1} - 2)\eta' - \tfrac{1}{2}(p_{12}u + p_{21}u^{-1})\zeta' \\
&\quad + \tfrac{1}{4}(p_{12}u + p_{21}u^{-1} + 2q_{12}u - 2q_{21}u^{-1} + 2q)\eta \\
&\quad - \tfrac{1}{4}(p_{12}u - p_{21}u^{-1} + 2q_{12}u + 2q_{21}u^{-1} - 2q_{11} - 2q_{22} - 1)\zeta = 0.
\end{aligned} \tag{11}
$$

Now while on each quadric given by (3), and hence on quadric (10), there lie two asymptotic tangents of R, it does not follow that there will be two asymptotic curves of R included in each one-parameter family. If two asymptotic curves of a ruled surface are chosen for directrix curves, the defining system of differential equations is characterized by the absence of certain terms of the first order.* In order that the two curves given by (11) be asymptotic curves it would be necessary for both the coefficient of ζ' in the first and that of η' in the second to vanish; but this is impossible. If either vanishes without the other, one of the curves is asymptotic. A more interesting situation develops when we impose a slightly greater restriction on the coefficients in (11). We shall first assume that the coefficients of both η' and ζ' in the first equation of (11) vanish, that is

$$
p_{12}u + p_{21}u^{-1} = 0, \qquad p_{12}u - p_{21}u^{-1} + 2 = 0. \tag{12}
$$

This makes C_η an asymptotic curve of R.

From (12), since $u = ae^z$, $p'_{12} = 2q_{12}$, $p'_{21} = 2q_{21}$, we find by differentiation that

$$
\begin{aligned}
p_{12}u - p_{21}u^{-1} + 2q_{12}u + 2q_{21}u^{-1} &= 0, \\
p_{12}u + p_{21}u^{-1} + 2q_{12}u - 2q_{21}u^{-1} &= 0.
\end{aligned} \tag{13}
$$

Making use of (12) and (13), we may now write system (11) as

$$
\begin{aligned}
\eta'' + \tfrac{1}{4}(2q_{11} + 2q_{22} + 1)\eta + \tfrac{1}{2}q\zeta &= 0, \\
\zeta'' - 2\eta' + \tfrac{1}{2}q\eta + \tfrac{1}{4}(2q_{11} + 2q_{22} + 1)\zeta &= 0.
\end{aligned} \tag{14}
$$

Equations (12) place certain restrictions on the coefficients of system (R). We find that $ap_{12} = -e^{-z}$, $p_{21} = ae^z$ and hence $2aq_{12} = e^{-z}$, $2q_{21} = ae^z$. Assumptions (12) have thus particularized four of the six coefficients of (R) and have at the same time introduced an arbitrary constant a. But this is not an essential constant, for the transformation $z = a\bar{z}$, which has no geometric significance, removes the a and leaves system (R) in the form

* Wilczynski, op. cit., p. 142.

$$(15)\qquad y'' - e^{-z}z' + q_{11}y + \tfrac{1}{2}e^{-z}z = 0, \quad z'' + e^{z}y' + \tfrac{1}{2}e^{z}y + q_{22}z = 0.$$

If now in (11) we replace p_{12}, p_{21}, $2q_{12}$, $2q_{21}$ with the respective values $-e^{-z}$, e^{z}, e^{-z}, e^{z}, we obtain the system of differential equations satisfied by η, ζ in the form

$$(16)\qquad \begin{aligned} &\eta'' - \tfrac{1}{2}(a-1/a)\eta' + \tfrac{1}{2}(a+1/a-2)\zeta' + \tfrac{1}{4}(2q_{11}+2q_{22}+1)\eta + \tfrac{1}{2}q\zeta = 0,\\ &\zeta'' - \tfrac{1}{2}(a+1/a+2)\eta' + \tfrac{1}{2}(a-1/a)\zeta' + \tfrac{1}{2}q\eta + \tfrac{1}{4}(2q_{11}+2q_{22}+1)\zeta = 0, \end{aligned}$$

and when $a=1$, (16) reduces to (14). It may be verified that *the directrix curves employed in system* (14) *constitute the two branches of the involute curve of that special type of surface R defined by the flecnode system* (15).

If, proceeding as above, we equate to zero the coefficients of η' and ζ' in the second equation of (11), system (11) reduces to

$$(17)\qquad \begin{aligned} \eta'' - 2\zeta' + \tfrac{1}{4}(2q_{11} + 2q_{22} + 1)\eta + \tfrac{1}{2}q\zeta = 0,\\ \zeta'' + \tfrac{1}{2}q\eta + \tfrac{1}{4}(2q_{11} + 2q_{22} + 1)\zeta = 0, \end{aligned}$$

so that C_ζ is now an asymptotic curve. At the same time system (R) reduces to a form which is identical with (15) after use has been made of the transformation $z=-a\bar{z}$. Equation (17) is recognized as that special case of (16) for which $a=-1$.

That the exponential coefficients of (15) do not occur in (16) is not surprising when it is seen that the transformation $z=e^{z}\bar{z}$, which leaves the directrix curves unchanged, reduces (15) to the form

$$(18)\qquad y'' - \bar{z}' + q_{11}y - \tfrac{1}{2}\bar{z} = 0, \quad \bar{z}'' + y' + 2\bar{z}' + \tfrac{1}{2}y + (q_{22}+1)\bar{z} = 0,$$

in which the only variable coefficients are q_{11} and q_{22}.

For system (15) we find that

$$(19)\qquad \rho = 2y' - e^{-z}z, \qquad \sigma = 2z' + e^{z}y.$$

By successive differentiation and the use of (15) and (19) we get

$$(20)\qquad \begin{aligned} \rho'' - e^{-z}\sigma' + q_{11}\rho + \tfrac{1}{2}e^{-z}\sigma &= -2q_{11}'y + \tfrac{1}{2}(2q_{11} + 2q_{22} + 1)e^{-z}z,\\ \sigma'' + e^{z}\rho' + \tfrac{1}{2}e^{z}\rho + q_{22}\sigma &= -\tfrac{1}{2}(2q_{11} + 2q_{22} + 1)e^{z}y - 2q_{22}'z. \end{aligned}$$

If now we impose upon system (15) the further conditions that q_{11} and q_{22} are unequal constants but are such that

$$(21)\qquad 2q_{11} + 2q_{22} + 1 = 0,$$

then (20) discloses the fact that the curves C_ρ, C_σ are the two branches of the flecnode curve of a ruled surface projectively equivalent to R. Wilczynski named the surface generated by the line $l_{\rho\sigma}$ the derivative surface of R.

The two sheets F_1, F_2 of R's flecnode surface are generated by the flecnode tangents l_{yp} and l_{zs}. On the first sheet C_y figures as one branch of its flecnode curve, and on the second sheet C_z plays the same role. The points given by the expressions $\theta = -e^z y$, $\phi = \rho + y$ are the flecnode points of l_{yp}. By differentiation, making use of (15), (19), and (21), we find that

$$(22)\quad \phi'' - e^{-z}\theta' + q_{11}\phi + \tfrac{1}{2}e^{-z}\theta = 0, \quad \theta'' + e^z\phi' + \tfrac{1}{2}e^z\phi + q_{22}\theta = 0.$$

The points given by the expressions $\xi = e^{-z}z$, $\psi = \sigma - z$, are the flecnode points on l_{zs}. Proceeding as above we find that

$$(23)\quad \xi'' - e^{-z}\psi' + q_{11}\xi + \tfrac{1}{2}e^{-z}\psi = 0, \quad \psi'' + e^z\xi' + \tfrac{1}{2}e^z\xi + q_{22}\psi = 0.$$

Because of the identity of forms of (15), (22), and (23), it follows that for the particular type of ruled surface which is defined by (15) and for which

$$q_{11} = c_1 \neq c_2 = q_{22}, \qquad 2q_{11} + 2q_{22} + 1 = 0,$$

the two sheets of its flecnode surface are projectively equivalent to it.

The assumptions on q_{11} and q_{22} reduce (16) to the form

$$(24)\quad \begin{aligned} &\eta'' - \tfrac{1}{2}(a - 1/a)\eta' + \tfrac{1}{2}(a + 1/a - 2)\zeta' + \tfrac{1}{2}q\zeta = 0, \\ &\zeta'' - \tfrac{1}{2}(a + 1/a + 2)\eta' + \tfrac{1}{2}(a - 1/a)\zeta' + \tfrac{1}{2}q\eta = 0; \end{aligned}$$

and from (24) by successive differentiation and elimination we obtain the fourth order differential equations defining the curves C_η, C_ζ of the one-parameter family defined by (24). These equations are found to be identical in form and free from the arbitrary constant a, that for C_η being $\eta^{(4)} + q\eta' - \frac{1}{4}q^2\eta = 0$. It follows that the curves of this family are all projectively equivalent, and since $q = \text{const.}$, they are anharmonic curves.*

To sum up, we have demonstrated the existence of a one-parameter family of ruled surfaces defined by a system of differential equations

$$y'' - e^{-z}z' + q_{11}y + \tfrac{1}{2}e^{-z}z = 0, \quad z'' + e^z y' + \tfrac{1}{2}e^z y + q_{22}z = 0,$$
$$q_{11} = c_1 \neq c_2 = q_{22}, \qquad 2q_{11} + 2q_{22} + 1 = 0,$$

for each of which the derivative surface as well as each sheet of the flecnode surface is projectively equivalent to the given surface, and on which there lies a one-parameter involution family of curves whose double elements are the two branches of the flecnode curve, all curves of the family being projectively equivalent anharmonic curves.

THE UNIVERSITY OF WASHINGTON

* Wilczynski, op. cit., p. 279.

CONCERNING CONTINUA IN A SEPARABLE SPACE WHICH DO NOT CROSS*

O. H. HAMILTON

In working with collections of continua it is sometimes useful to know something of the character of the point set consisting of all the points common to two or more members of the collection. Also it is of advantage to know conditions under which we may subtract a countable number of continua from the collection and have left a collection of mutually exclusive continua. The theorem which we shall prove may aid in answering questions of this nature.

All definitions and discussions will refer to point sets in a connected and locally connected separable space S.

DEFINITION 1. *If g_1 and g_2 are two continua each of which separates S, g_1 will be said to cross g_2 provided there exist two complementary domains of g_2 (maximum connected domains of $S-g_2$) which contain points of g_1.*

DEFINITION 2. *If* (1) M_1, M_2, *and* M_3 *are three point sets in* S, (2) g_1 *and* g_2 *are two continua in* S, (3) M_1 *is in a complementary domain* D_1 *of* g_1 *which does not contain a point of* g_2, (4) M_2 *is in a complementary domain* D_2 *of* g_2 *which does not contain a point of* g_1, *and* (5) M_3 *is in a complementary domain* D_3 *of* g_1+g_2 *distinct from* D_1 *or* D_2, *then* g_1+g_2 *will be said to separate* M_1, M_2, *and* M_3 *symmetrically with respect to* M.

LEMMA 1. *If* (1) g_1 *and* g_2 *are two continua each of which separates* S, (2) *each separates some complementary domain of the other, and* (3) *neither crosses the other, then there exist domains* D_1, D_2, *and* D_3 *such that* g_1+g_2 *separates* D_1, D_2, *and* D_3 *symmetrically with respect to* D_3.

PROOF. $S-g_1$ is the sum of two mutually separated point sets S_1 and S_2. One of these, say S_1, is such that the continuum g_2 is a subset of S_1+g_1. Let D_1 be a maximum connected domain of S_2. Also $S-g_2$ is the sum of two mutually separated point sets S_3 and S_4. One of these, say S_3, is such that g_1 is a subset of S_3+g_2. Let D_2 be a maximum connected domain of S_4. Let D_3 be a maximum connected domain of g_1+g_2 distinct from D_1 or D_2. We know D_3 exists, since by hypotheses each of the continua g_1 and g_2 separates some complementary domain of the other. The domains D_1, D_2, and D_3 are then separated by g_1+g_2 symmetrically with respect to D_3.

* Presented to the Society, April 16, 1938.

LEMMA 2. *If g and h are two continua which do not cross and which have no complementary domain in common, and if D is a complementary domain of g such that h contains a point of D and is a subset of $\overline{D}$, then $S-\overline{D}$ lies in a connected complementary domain of h.*

PROOF. Suppose D_1 and D_2 are two distinct complementary domains of h, each of which contains a point of $S-\overline{D}$. One of these domains does not contain a point of g, since g and h do not cross. Suppose D_2 contains no point of g. Then D contains no point of D_2 and therefore contains no point of the boundary of D_2. Since, by hypothesis, every point of h is in $D+g$, it follows that every point of the boundary of D_2 is in g as well as in h, and therefore is in $g \cdot h$. Since we have supposed that g contains no point of D_2, h and g have the complementary domain D_2 in common, and this contradicts a hypothesis of the lemma. We arrive at a similar contradiction if we suppose that g contains no point of D_1.

Notation. In what follows, if h_1 and h_2 are any two continua whose sum separates the points P_1, P_2, and P_3 symmetrically with respect to P_3, then D_{h1} and D_{h2} will designate the complementary domains of h_1+h_2 which contain P_1 and P_2, respectively. A similar notation will be used if h is replaced by any other symbol.

LEMMA 3. *If* (1) g_1, g_2, h_1, *and* h_2 *are four continua no one of which crosses another or has a complementary domain in common with another one of the four,* (2) *both* g_1+g_2 *and* h_1+h_2 *separate the points* P_1, P_2, *and* P_3 *symmetrically with respect to* P_3, *and* (3) $h_1 \cdot h_2$ *contains a point not in* g_1+g_2, *then the domain* $S_1=S-(h_1+h_2+D_{h1}+D_{h2})$ *is a proper subset of the domain* $S_2=S-(g_1+g_2+D_{g1}+D_{g2})$ *and* h_1+h_2 *contains* $g_1 \cdot g_2$.

PROOF. By hypothesis (3), $h_1 \cdot h_2$ contains a point P of some complementary domain of g_1+g_2. We know P cannot be in D_{g1}, for if so, then by Lemma 2, $S-\overline{D}_{g1}$, and therefore P_2+P_3, lies in a connected complementary domain E_2 of h_2. Since h_1+h_2 separates P_1, P_2, and P_3 symmetrically with respect to P_3, $\overline{E}_2$ does not contain h_1 (see Definition 2). Let E_1 designate the complementary domain of h_2 which with its boundary contains h_1. By Lemma 2, $S-\overline{E}_1$, which contains E_2, lies in a connected complementary domain E_3 of h_1. That is, the complementary domain E_2 of h_2 which contains P_2+P_3 is a subset of the complementary domain E_3 of h_1. Then h_1+h_2 does not separate P_2 from P_3, and we have a contradiction of a hypothesis of the lemma. It follows that $h_1 \cdot h_2$ cannot contain a point of D_{g1} and similarly cannot contain a point of D_{g2}.

Then $h_1 \cdot h_2$ must contain a point of S_2. Since D_{g1} and D_{g2} contain

P_1 and P_2, respectively, and since h_1 separates P_1 from P_2 and contains no point of $D_{g1}+D_{g2}$, it follows that D_{g1} and D_{g2} lie in mutually exclusive complementary domains of h_1. There are points Q_1 and Q_2 on the boundaries of D_{g1} and D_{g2} which are in g_1 and g_2, respectively, but which are not points of h_1, since h_1 and g_1, and similarly h_1 and g_2, have no complementary domain in common. Then there exist mutually exclusive complementary domains F_1 and F_2 of h_1 such that g_1 and g_2 are subsets of $\overline{F}_1$ and $\overline{F}_2$, respectively. Consequently $g_1 \cdot g_2$ is a subset of $\overline{F}_1 \cdot \overline{F}_2$, that is, of h_1. Therefore, as required, h_1+h_2 contains $g_1 \cdot g_2$. By hypothesis, D_{h1} and D_{h2} contain P_1 and P_2, respectively, and the sets $\overline{D}_{h1}$ and $\overline{D}_{h2}$ contain $D_{g1}+g_1$ and $D_{g2}+g_2$, respectively. Therefore S_1 is a subset of S_2; and S_1 is a proper subset of S_2, since by assumption S_2 contains a point of $h_1 \cdot h_2$.

THEOREM. *Let G be a collection of continua having the following properties*: (1) *No continuum of G crosses or contains a complementary domain in common with any other continuum of the collection.* (2) *Each continuum of G separates S and also separates some complemenetary domain of each other continuum in the collection. Then there exists a countable collection H of G such that each point common to two continua of the collection G is contained in some continuum of the collection H.**

PROOF. Since S is separable, there exists a countable subset K of S which has the property that every point of S is a point of K or a limit point of K. Let L be a countable collection of the permutations of the points of K taken three at a time. Let p be any permutation $P_1P_2P_3$ of the collection L. Let W_p be the collection of all sets each of which consists of two continua a and b of the collection G such that $a \cdot b$ is not vacuous and such that $a+b$ separates P_1, P_2, and P_3 symmetrically with respect to P_3. In accordance with the notation used above, if $w_\theta = a_\theta + b_\theta$ is any set of the collection W_p, then $D_{\theta 1}$ and $D_{\theta 2}$ will designate the complementary domains of $a_\theta+b_\theta$ which contain P_1 and P_2, respectively, and S_θ will designate the domain $S-(D_{\theta 1}+D_{\theta 2}+a_\theta+b_\theta)$. A similar notation will be used if w_θ is replaced by any similar symbol. Let α be a well ordered sequence of the sets of the collection W_p. Let β be a well ordered subsequence $v_1, v_2, \cdots, v_\omega, v_{\omega+1}, \cdots$ of α. If $v_\theta = w_\phi$, let c_θ and d_θ designate a_ϕ and b_ϕ, respectively. We shall define v_1 as being the first element of α. Let v_2 be the first element of α which follows v_1 and has the property

* That the set of points each of which is common to two members of the collection G is not necessarily countable may be seen if, in the cartesian plane, we define G as being the collection of all continua, each of which is the sum of the X axis between two points A and B and two vertical rays extending upward from A and B.

that $c_2 \cdot d_2$ contains a point not in c_1+d_1. In general, if γ is any ordinal number such that v_ψ is defined for each ordinal number ψ which precedes γ, let v_γ be the first element of α which follows every v_ψ and has the property that $c_\gamma \cdot d_\gamma$ contains a point not in any $c_\psi+d_\psi$. By Lemma 3 it follows that if θ and ϕ are any two ordinal numbers such that θ precedes ϕ, then $c_\phi+d_\phi$ contains $c_\theta \cdot d_\theta$ and S_ϕ is a proper subset of S_θ. Furthermore if w_τ is any element whatever of the sequence α, then by virtue of the definition of the subsequence β, it follows that $a_\tau \cdot b_\tau$ is a subset of some set of the sequence β. By a well known property of abstract sets, since S is separable, it follows that the sequence β is countable. Let H_p designate the countable collection of continua

$$c_1, d_1, c_2, d_2, \cdots, c_\omega, d_\omega, c_{\omega+1}, d_{\omega+1}, \cdots.$$

Let H be the collection of all continua of the collection G which are in H_p for some permutation p of the collection L. Then since H_p is countable for each p and since L is countable, it follows that H is countable.

Now let h_1 and h_2 be any two elements of the collection G which have a point in common. Since S is separable and locally connected, it follows from Lemma 1 that there exists a permutation p, $P_1P_2P_3$, of L such that h_1+h_2 separates P_1, P_2, and P_3 symmetrically with respect to P_3. Consequently h_1+h_2 belongs to W_p and $h_1 \cdot h_2$ is a subset of some continuum of G which belongs to H_p and therefore is a subset of some continuum of G which belongs to the countable collection H. The theorem is therefore true.

Corollaries. The following are obvious corollaries to the theorem:

COROLLARY 1. *If G is any collection of continua satisfying the hypotheses of the theorem, then there exists a subcollection G' of mutually exclusive continua of G which contains all but a countable number of the continua of G.*

COROLLARY 2. *If G is a collection of continua in the euclidean plane, each of which is an open curve or a simple closed curve, and if no one of these continua crosses any other one of the collection, then there exists a countable subcollection H of G such that every point common to two elements of the collection G is in some element of the collection H.*

Examples. We see that the condition that the elements of G do not cross is necessary if we let G consist of all straight lines in the euclidean plane. It is to be noted that under these circumstances no two elements of G have a complementary domain in common but

each two nonparallel elements of G cross each other. Obviously the conclusions of the theorem do not hold.

The following example will show that the condition that no two elements of the collection G shall have a complementary domain in common is also necessary. In the cartesian plane let M be a circle of radius 1 and center at the origin, and N a circle of radius 1 and center at the point (5, 5). Let G_1 be a collection which contains each continuum which is the sum of M and a horizontal straight line interval of length 10 whose left-hand end point is on the circle M and which contains no point within M. Let G_2 be a collection which contains each continuum which is the sum of N and a vertical straight line interval of length 10 whose upper end point is on the circle N and which contains no point within N. Let $G=G_1+G_2$. No element of G crosses any other element of G, but uncountably many have a complementary domain in common with some other element of the collection. However, it is evident that no countable subcollection of G covers the set of points each of which is common to two continua of the collection G.

It is not known whether or not the condition that each element of G shall separate some complementary domain of every other one can be omitted.

OKLAHOMA AGRICULTURAL AND MECHANICAL COLLEGE

A PRINCIPAL AXIS TRANSFORMATION FOR NON-HERMITIAN MATRICES

CARL ECKART AND GALE YOUNG

The availability of the principal axis transformation for hermitian matrices often simplifies the proof of theorems concerning them. In working with non-hermitian matrices (square or rectangular) it was found that a generalization of this transformation has a similar use for them.* A special case of this generalization has been investigated by Sylvester† who proved Theorem 1 (below) for square matrices with real elements. The unitary matrices U and V are in that case orthogonal matrices with real elements. Special cases had also been

* C. Eckart, *The kinetic energy of polyatomic molecules*, Physical Review, vol. 46 (1934), p. 383; C. Eckart and G. Young, *The approximation of one matrix by another of lower rank*, Psychometrika, vol. 1 (1936), p. 211; A. S. Householder and G. Young, *Matrix approximation and latent roots*, American Mathematical Monthly, vol. 45 (1938), p. 302.

† Sylvester, Messenger of Mathematics, vol. 19 (1889), p. 42.

discussed earlier by Beltrami and Jordan, and more recently Autonne and E. T. Browne have proved the theorem for square matrices with complex elements.†

The following definitions will be convenient for the present purpose. An (r, s) matrix is one having r rows and s columns; its elements may be complex numbers. The hermitian transpose of an (r, s) matrix A, whose elements are a_{ij}, is the (s, r) matrix A^* whose elements are $(a^*)_{ji} = \bar{a}_{ij}$. An (r, s) matrix is diagonal if its elements a_{ij} are all zero unless $i = j$.

THEOREM 1. *For every (r, s) matrix A, there are two unitary matrices U and V, such that*

$$D = U^*AV$$

is a diagonal matrix with real elements, none of which are negative.

The proof of this theorem may be based on the observation that AA^* is a non-negative definite hermitian (r, r) matrix; for it is the Gram matrix of the rows of A, considered as vectors. Consequently there are r vectors (that is, r $(r, 1)$ matrices) X_i such that

$$AA^*X_i = d_i^2 X_i \tag{1}$$

and

$$X_i^*X_k = \delta_{ik}, \qquad i, k = 1, \cdots, r. \tag{2}$$

The numbers d_i^2 are the characteristic values of AA^*, and the X_i are unit vectors along its principal axes. The numbers d_i are real and may be defined to be nonnegative. It is convenient to arrange the numbering of these vectors so that

$$d_1 \geqq d_2 \geqq \cdots \geqq d_n > 0, \qquad d_{n+1} = \cdots = d_r = 0. \tag{3}$$

In the same way, there are s vectors (that is, s $(s, 1)$ matrices) Y_j such that

$$A^*AY_j = e_j^2 Y_j \tag{4}$$

and

$$Y_j^*Y_l = \delta_{jl}. \tag{5}$$

If

$$e_1 \geqq e_2 \geqq \cdots \geqq e_m > 0, \qquad e_{m+1} = \cdots = e_s = 0,$$

† Autonne, *Sur les matrices hypohermitiennes et les unitaires*, Comptes Rendus de l'Académie des Sciences, Paris, vol. 156 (1913), pp. 858–860; E. T. Browne, this Bulletin, vol. 36 (1930), p. 707.

it can be shown that $m=n$ and $e_i=d_i$ whenever $i \leqq n$. For if X_i are the vectors of (1), (2), and (3), then the n vectors defined by

$$Y_i = A^* X_i / d_i, \qquad i \leqq n, \tag{6}$$

will satisfy (4) and (5) with $e_i=d_i$. Since the characteristic values are unique, it follows that m cannot be less than n; inverting the argument, we see that n cannot be less than m.

Any set of vectors X_i for which (1), (2), and (3) hold may be considered as the columns of a unitary (r, r) matrix U. Then let (6) define the first n columns of an (s, s) matrix V, and fill in the remaining columns to make V unitary. These matrices U and V then satisfy the requirements of the theorem.

To prove this we may first observe that if D is the matrix U^*AV, then $DD^* = U^*AA^*U$ is a diagonal matrix with diagonal elements d_i^2, and $D^*D = V^*A^*AV$ is a diagonal matrix with diagonal elements e_i^2. Furthermore, if the matrix D is written as

$$D = \left\| \begin{matrix} D_1 & D_2 \\ D_3 & D_4 \end{matrix} \right\|,$$

where D_1, D_2, D_3, and D_4 are (n, n), $(n, s-n)$, $(r-n, n)$, and $(r-n, s-n)$ matrices, respectively, then these properties of DD^* and D^*D imply

$$D_3D_3^* + D_4D_4^* = 0, \qquad D_2^*D_2 + D_4^*D_4 = 0$$

(among other equations). Since $D_2^*D_2$, $D_3D_3^*$, and $D_4D_4^*$ are all non-negative definite hermitian matrices, it follows that they are all null matrices, and from this, that D_2, D_3, and D_4 are all null matrices. It remains to be shown that D_1 is diagonal with no negative elements. Its ij-element may be written $X_i^*AY_j$; and from (6), (1), and (2) it readily follows that

$$X_i^*AY_j = d_j\delta_{ij}, \qquad i, j \leqq n,$$

which completes the proof of Theorem 1.

The above also proves the following result:

COROLLARY. *In Theorem* 1, U^* *may be any unitary matrix which diagonalizes* AA^*, *and there then exists a unitary matrix* V *such that the theorem is true. Similarly,* V^* *may be taken as any unitary matrix which diagonalizes* A^*A, *and there then exists a matrix* U *satisfying the requirements of the theorem.*

The theorem on the simultaneous transformation of two hermitian matrices to principal axes also generalizes.

THEOREM 2. *If A and B are both (r, s) matrices, then there are two unitary matrices U and V such that $E = U^*AV$, and $F = U^*BV$ are both diagonal matrices with real elements and such that E has no negative elements, if and only if AB^* and B^*A are both hermitian matrices.*

The necessity of the condition is an immediate consequence of the invariance of the class of hermitian matrices under transformations of the form UCU^* when U is unitary, for EF^* is hermitian and $AB^* = UEF^*U^*$, and so on. The sufficiency may be proved as follows. Because of Theorem 1, it is no loss of generality to suppose that A has already been transformed to the form

$$\begin{Vmatrix} D & O_2 \\ O_3 & O_4 \end{Vmatrix},$$

where D is a real diagonal (n, n) matrix of rank n, having no negative elements, and O_2, O_3, O_4 are null matrices. The matrix B may be divided into corresponding submatrices:

$$\begin{Vmatrix} G & K \\ L & H \end{Vmatrix}.$$

Then the condition of the theorem leads to

$$K = O_2, \qquad L = O_3, \qquad DG^* = GD, \qquad G^*D = DG.$$

In element notation, the last two equations are

$$d_i\bar{g}_{ji} = d_jg_{ij}, \qquad d_i\bar{g}_{ji} = d_jg_{ij}.$$

Since d_i, $d_j > 0$, it readily follows that G is hermitian and that $DG = GD$. From the theorem on the simultaneous transformation of two hermitian matrices to principal axes, it follows that there is a unitary (n, n) matrix P such that $P^*DP = D$ and P^*GP is diagonal with real elements. From Theorem 1, it follows that there are also two unitary matrices Q and R such that Q^*HR is a diagonal matrix with real elements. It is then readily seen that the matrices

$$U = \begin{Vmatrix} P & O_3^* \\ O_3 & Q \end{Vmatrix}, \qquad V = \begin{Vmatrix} P & O_2 \\ O_2^* & R \end{Vmatrix}$$

will satisfy Theorem 2.

The authors desire to acknowledge helpful discussion with Professor G. A. Bliss and Dr. Saunders MacLane.

THE UNIVERSITY OF CHICAGO

METRIC SPACES WITH GEODESIC RICCI CURVES. II.

JACK LEVINE

1. **Introduction.** In this paper we give a partial classification of four-dimensional metric spaces admitting geodesic Ricci curves. The results of I* will be assumed known, along with the notations of that paper.

We assume a given set of independent vectors† λ^i_a such that $c^i_{ij}=0$ (i not summed), so as to obtain geodesic curves, and impose conditions (24), (25), (26) of I on the θ_a. From I, (25) we see that if $\mu_{kk}=0$ for any k, then by I, (24), we have (since $|\lambda^i_a| \neq 0$) $\partial\mu_k/\partial x^a=0$, or $\mu_k=$const. If, however, for any given k, $\mu_{kk}\neq 0$, then from I, (25), $c^k_{ij}=0$, for all i and j. This gives us a means of classifying the spaces according to the number of μ_{kk} which equal zero. For $n=4$ there are five cases, which, without loss of generality, we may take in the form:

(A) $\mu_{ii} \neq 0$; (B) $\mu_{11}=0$; $\mu_{22}, \mu_{33}, \mu_{44} \neq 0$;

(C) $\mu_{11}=\mu_{22}=0$; $\mu_{33}, \mu_{44} \neq 0$; (D) $\mu_{11}=\mu_{22}=\mu_{33}=0$; $\mu_{44} \neq 0$;

(E) $\mu_{ii}=0$.

In the following discussion we consider cases (A), (B), and certain special cases under (C). For these special cases we shall merely state the results.

2. **Cases** (A) **and** (B). For case (A) we see from I, (25) that $c^k_{ij}=0$, which implies that V_4 is a flat space.

We consider now case (B). Here μ_1 and hence θ_1 is constant. From I, (25) we have

$$c^2_{ij}=c^3_{ij}=c^4_{ij}=0. \tag{1}$$

If in I, (26) we make the substitution

$$\bar{c}^i_{jk}=\frac{\theta_j\theta_k}{\theta_i}c^i_{jk}, \tag{2}$$

we obtain I, (23) in the barred quantities. We call this resulting equation I, (23′).

* *Metric spaces with geodesic Ricci curves*, I, this Bulletin, vol. 44 (1938), pp. 145–152. We refer to this paper as I, and the notation I, (23), for example, refers to its equation (23).

† All indices take the values 1, 2, 3, 4 unless otherwise noted.

Since by (2), whenever $c^4_{jk}=0$ the corresponding $\bar{c}^4_{jk}$ are also zero, we may solve I, (23′) directly for the $\bar{c}$'s, the normalized form of the c's, without the necessity of first determining the θ's. This process will be made clearer in what follows.

By the use of (1) and (2), I, (23′) reduces to

$$(3)\qquad \bar{c}^1_{24}\bar{c}^1_{34}=0,\ \bar{c}^1_{23}\bar{c}^1_{43}=0,\ \bar{c}^1_{32}\bar{c}^1_{42}=0,\ \bar{\Delta}_4\bar{c}^1_{34}=0,\ \bar{\Delta}_3\bar{c}^1_{34}=0.$$

We may thus take, for example, $\bar{c}^1_{23}=\bar{c}^1_{24}=0$, and there remains to be determined $\bar{c}^1_{34}$, the only nonzero $\bar{c}$.

A set of operators $\bar{\Delta}_a$ satisfying the relations

$$(4)\qquad (\bar{\Delta}_a,\bar{\Delta}_b)=\bar{c}^k_{ab}\bar{\Delta}_k,$$

where

$$\bar{\Delta}_a\equiv\bar{\lambda}^i_{a|}\frac{\partial}{\partial x^i},$$

with all $\bar{c}^k_{ab}=0$ except $\bar{c}^1_{34}$, can be expressed in the form

$$(5)\qquad \bar{\Delta}_\alpha=\frac{\partial}{\partial x^\alpha},\qquad \bar{\Delta}_4=A(x^3,x^4)\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^4},\qquad \alpha=1,2,3;$$

whence $\bar{c}^1_{34}=\partial A/\partial x^3$.

From the last two equations of (3) and (5) we see that $\bar{c}^1_{34}=k$ =const. Hence $A=kx^3+G(x^4)$, and by the transformation

$$x'^1=x^1-\int G dx^4,\qquad x'^\alpha=x^\alpha,\qquad \alpha=2,3,4,$$

we can make $G=0$. As the canonical form for the $\bar{\Delta}$'s we have then (dropping primes)

$$\bar{\Delta}_\alpha=\frac{\partial}{\partial x^\alpha},\qquad \bar{\Delta}_4=kx^3\frac{\partial}{\partial x^1}+\frac{\partial}{\partial x^4},\qquad \alpha=1,2,3,$$

where $k\neq 0$.

To obtain the θ_a we use

$$\Delta_{j\mu_i}=\frac{1}{\theta_j}\bar{\Delta}_{j\mu_i}=0,\quad \text{or}\quad \bar{\Delta}_{j\mu_i}=0,\qquad i\neq j.$$

This gives

$$\theta_\alpha=\theta_\alpha(x^\alpha),\quad \alpha=2,3,4;\qquad \theta_1=\text{const.}$$

If we substitute these values for the θ_i in I, (26) and use (2) to ob-

tain c^1_{34}, we will find that I, (26) are satisfied identically. Hence the θ_i, $(i=2, 3, 4)$, are arbitrary functions of their respective arguments. The forms of the $\bar{\lambda}^i_{a|}$ and g_{ij} are given in the last section.

3. **Case** (C). In this case θ_1 and θ_2 are constant, and $c^3_{ij}=c^4_{ij}=0$. Hence, by (2),

$$\bar{c}^3_{ij} = \bar{c}^4_{ij} = 0.$$

It is possible to find a coordinate system in which the $\bar{\Delta}_i$ assume the canonical form

$$\bar{\Delta}_1 = \frac{\partial}{\partial x^1}, \qquad \bar{\Delta}_2 = \frac{\partial}{\partial x^2}, \qquad \bar{\Delta}_3 = \bar{\lambda}^1_{3|}\frac{\partial}{\partial x^1} + \bar{\lambda}^2_{3|}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3},$$

(6)

$$\bar{\Delta}_4 = \bar{\lambda}^1_{4|}\frac{\partial}{\partial x^1} + \bar{\lambda}^2_{4|}\frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^4},$$

and from (4) we find

(7) $$\bar{\lambda}^1_{3|}(234), \qquad \bar{\lambda}^2_{3|}(134), \qquad \bar{\lambda}^1_{4|}(234), \qquad \bar{\lambda}^2_{4|}(134),$$

where $f(ij\cdots k)$ means that f is a function of $x^i, x^j, \cdots, x^k$.

For convenience we shall drop the bars on the $\bar{\lambda}$ and $\bar{c}$ and then use the notations

$$a = c^1_{23}, \quad b = c^1_{24}, \quad c = c^1_{34}, \quad d = c^2_{13}, \quad f = c^2_{14}, \quad g = c^2_{34},$$
$$\alpha = \lambda^1_{3|}, \quad \beta = \lambda^2_{3|}, \quad \gamma = \lambda^1_{4|}, \quad \delta = \lambda^2_{4|}.$$

Then from (4) and (7) we obtain,

$$a = \frac{\partial\alpha}{\partial x^2}, \qquad b = \frac{\partial\gamma}{\partial x^2}, \qquad f = \frac{\partial\delta}{\partial x^1}, \qquad d = \frac{\partial\beta}{\partial x^1},$$

(8) $$c = \beta\frac{\partial\gamma}{\partial x^2} - \delta\frac{\partial\alpha}{\partial x^2} + \frac{\partial\gamma}{\partial x^3} - \frac{\partial\alpha}{\partial x^4}, \quad g = \alpha\frac{\partial\delta}{\partial x^1} - \gamma\frac{\partial\beta}{\partial x^1} + \frac{\partial\delta}{\partial x^3} - \frac{\partial\beta}{\partial x^4};$$

$$a(234),\ b(234),\ c(1234),\ d(134),\ f(134),\ g(1234),\ \alpha(234),$$
$$\beta(134),\ \gamma(234),\ \delta(134).$$

If in I, (23′) we take for the indices bc the values 34, we obtain

$$af + bd + e_1e_2(ab + df) = (f + be_1e_2)(d + ae_1e_2) = 0.$$

Let us take

(9) $$d = -e_1e_2a,$$

which, from (8), shows that $d(34)$, $a(34)$. The remaining five equa-

tions of I, (23′) now have the following form, use being made of (6), (9), and the equation $\Delta_2 a = 0$ which follows from (6):

$$\gamma \frac{\partial c}{\partial x^1} + \delta \frac{\partial c}{\partial x^2} + \frac{\partial c}{\partial x^4} = - e_1 e_2 f g, \tag{10}$$

$$\gamma \frac{\partial g}{\partial x^1} + \delta \frac{\partial g}{\partial x^2} + \frac{\partial g}{\partial x^4} = - e_1 e_2 b c, \tag{11}$$

$$\alpha \frac{\partial c}{\partial x^1} + \beta \frac{\partial c}{\partial x^2} + \frac{\partial c}{\partial x^3} + e_2 e_3 \frac{\partial b}{\partial x^2} = a g, \tag{12}$$

$$\alpha \frac{\partial g}{\partial x^1} + \beta \frac{\partial g}{\partial x^2} + \frac{\partial g}{\partial x^3} + e_1 e_3 \frac{\partial f}{\partial x^1} = - e_1 e_2 a c, \tag{13}$$

$$e_1 \left(\delta \frac{\partial b}{\partial x^2} + \frac{\partial b}{\partial x^4} \right) + e_2 \left(\gamma \frac{\partial f}{\partial x^1} + \frac{\partial f}{\partial x^4} \right) = e_1 e_2 e_3 c g. \tag{14}$$

4. **Subcases of** (C). We shall consider several subcases under (C) and divide these according as $a = 0$ and $a \neq 0$:

(C 1) $a = 0$; (C 2) $a \neq 0$.

For (C 1), we have from (9), $d = 0$, and by a change of coordinates we can make $\alpha = \beta = 0$. If we then differentiate (10) and (11) with respect to x^1 and x^2, we find that

$$f \frac{\partial^2 c}{\partial x^2 \partial x^2} = 0, \qquad b \frac{\partial^2 g}{\partial x^1 \partial x^1} = 0. \tag{15}$$

This leads to the subcases under (C 1):

(C 1.1) $b = f = 0$; (C 1.2) $b \neq 0, f = 0$; (C 1.3) $b \neq 0, f \neq 0$.

The results for these three subcases are as follows:

Case (C 1.1). From (14) we see that c or g is zero. In either case we have case (B) repeated.

Case (C 1.2). For this case we obtain the following two possible solutions:

$$b_1 = k B_1', \; g_1 = B_1, \; c_1 = - \frac{e_1 e_2}{k} = c, \delta_1 = B_1 x^3, \gamma_1 = c x^3 + b_1 x^2, \tag{16}$$

$$b_2 = k B_2', \; g_2 = B_2, \; c_2 = c, \; \delta_2 = B_2 x^3, \; \gamma_2 = c x^3 + b_2 x^2, \tag{17}$$

where B_1 and B_2 are given by the relations

$$B_1 = k_1 \sin \frac{x^4}{k} + k_2 \cos \frac{x^4}{k}, \quad \text{if} \quad e_1 e_3 = +1,$$

$$B_2 = k_1 e^{x^4/k} + k_2 e^{-x^4/k}, \quad \text{if} \quad e_1 e_3 = -1, \quad k, k_1, k_2 \text{ const.}$$

To obtain the θ_i we proceed as in case (B) and find

$$\theta_1, \theta_2 \text{ const.}, \ \theta_3 = \theta_3(x^3), \theta_4 = \theta_4(x^4).$$

If we use (2) to obtain c^1_{24}, c^1_{34}, c^2_{34} (unbarred) and substitute in I, (26), these equations are satisfied identically; hence θ_3 and θ_4 are arbitrary. The forms of $\overline{\lambda}^i_{\alpha|}$ and g_{ij} for this case are given in the last section.

Case (C 1.3). It can be shown that we must have b, c, f, g all functions of x^4 only, and connected by the relations

$$(18) \qquad c' = -e_1 e_2 f g, \quad g' = -e_1 e_2 b c, \quad e_2 b' + e_1 f' = e_3 c g.$$

If either c or g is zero, then by (18) the other is also, and the third equation of (18) shows that

$$e_2 b + e_1 f = \iota = \text{const.},$$

and

$$\gamma = E(x^4)x^2, \qquad \delta = (\iota - e_2 E) e_1 x^1,$$

with $E(x^4)$ arbitrary.

As for (C 1.2), $\theta_3(x^3)$, $\theta_4(x^4)$ are arbitrary.

If neither c nor g is zero, we have (18) to determine b, c, f, g, none of which is now zero. By eliminating b and f we obtain

$$(19) \qquad e_1(g'/c)' + e_2(c'/g)' = -e_3 c g.$$

We may take, for example, g as arbitrary, and then determine b, c, f from (18) and (19).

Case (C 2). Here $a \neq 0$; whence follows $d \neq 0$. It can be shown that we must have $b \neq 0$, $f \neq 0$ also. We consider only the special case (C 2.1) in which c or g is zero, from which, by (10) or (11), it follows that the other is also. The results for this case, (C 2.1), are:

$$\alpha = a(x^3, x^4)x^2, \quad \beta = -e_1 e_2 a x^1, \quad \gamma = b(x^3, x^4)x^2, \quad \delta = -e_1 e_2 b x^4,$$

$$d = -e_1 e_2 a, \qquad f = -e_1 e_2 b,$$

and a and b are arbitrary subject to $\partial b/\partial x^3 = \partial a/\partial x^4$.

5. **Forms for the** g_{ij}. In this section we obtain the forms of the g_{ij} for the various cases previously considered. From

$$g^{ij} = \sum_h e_h \bar{\lambda}^i_{h|} \bar{\lambda}^j_{h|}$$

we can easily obtain the g_{ij}. The $\bar{\lambda}^i_{a|}$ are the normalized form of the Ricci congruence vectors and are obtained from $\bar{\Delta}_a = \bar{\lambda}^i_{a|} \partial/\partial x^i$. The canonical forms of the $\bar{\Delta}_a$ have been used for each case. We now use the bars on the λ's and the c's.

Case (A). This gives a flat space, and $\bar{\lambda}^i_{a|} = \delta^i_a$, $g_{ij} = e_i \delta^i_j$.

Case (B). For this case $\bar{c}^1_{34} = k = \text{const.}$ (nonzero), and the rest of the $\bar{c}$'s are zero. We have also $\theta_1 = k_1 = \text{const.}$, $\theta_\alpha(x^\alpha)$, $(\alpha = 2, 3, 4)$. The $\bar{\lambda}^i_{a|}$ and g_{ij} have, respectively, the following forms:

$$\bar{\lambda}^i_{a|}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ kx^3 & 0 & 0 & 1 \end{pmatrix}, \qquad g_{ij}: \begin{pmatrix} e_1 & 0 & 0 & -e_1 k x^3 \\ 0 & e_2 & 0 & 0 \\ 0 & 0 & e_3 & 0 \\ -e_1 k x^3 & 0 & 0 & e_1 (k x^3)^2 + e_4 \end{pmatrix}.$$

Case (C). We obtain $\bar{c}^3_{ij} = \bar{c}^4_{ij} = 0$, θ_1, θ_2 const., $\theta_3(x^3)$, $\theta_4(x^4)$.

Case (C 1). The forms of $\bar{\lambda}^i_{a|}$ and g_{ij} are the following:

$$\bar{\lambda}^i_{a|}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma & \delta & 0 & 1 \end{pmatrix}, \qquad g_{ij}: \begin{pmatrix} e_1 & 0 & 0 & -e_1\gamma \\ 0 & e_2 & 0 & -e_2\delta \\ 0 & 0 & e_3 & 0 \\ -e_1\gamma & -e_2\delta & 0 & e_1\gamma^2 + e_2\delta^2 + e_4 \end{pmatrix}.$$

Case (C 1.1). This case is equivalent to case (B).

Case (C 1.2). In this case we have $\bar{c}^1_{23} = \bar{c}^2_{13} = \bar{c}^2_{14} = 0$, $\bar{c}^1_{24} \neq 0$,

$$\gamma_1 = cx^3 + (k_1 \cos x^4/k - k_2 \sin x^4/k)x^2,$$

$$\delta_1 = (k_1 \sin x^4/k + k_2 \cos x^4/k)x^3, \qquad e_1 e_3 = +1,$$

$$\gamma_2 = cx^3 + (k_1 e^{x^4/k} - k_2 e^{-x^4/k})x^2, \qquad \delta_2 = (k_1 e^{x^4/k} + k_2 e^{-x^4/k})x^3, \qquad e_1 e_3 = -1,$$

$$(\bar{c}^1_{24})_1 = k_1 \cos x^4/k - k_2 \sin x^4/k, \qquad (\bar{c}^1_{34})_1 = c = -\frac{e_1 e_2}{k},$$

$$(\bar{c}^2_{34})_1 = k_1 \sin x^4/k + k_2 \cos x^4/k, \qquad (\bar{c}^1_{34})_2 = c = -\frac{e_1 e_2}{k},$$

$$(\bar{c}^1_{24})_2 = k_1 e^{x^4/k} - k_2 e^{-x^4/k}, \quad (\bar{c}^2_{34})_2 = k_1 e^{x^4/k} + k_2 e^{-x^4/k}, \quad k, k_1, k_2 \text{ const.}$$

Case (C 1.3). We have $\bar{c}^1_{23}=\bar{c}^2_{13}=0$, $\bar{c}^1_{24}\neq 0$, $\bar{c}^2_{14}\neq 0$. If $\bar{c}^1_{34}=\bar{c}^2_{34}=0$, then

$$\gamma = E(x^4)x^2, \qquad \delta = (t - e_2E)e_1x^1,$$

$$\bar{b} = \bar{c}^1_{24} = E(x^4), \qquad \bar{f} = \bar{c}^2_{14} = (t - e_2E)e_1, \quad E \text{ arbitrary}, \quad t = \text{const.}$$

If $\bar{c}^1_{34}\bar{c}^2_{34}\neq 0$, then $\bar{c}^1_{24}$, $\bar{c}^2_{14}$, $\bar{c}^1_{34}$, $\bar{c}^2_{34}$ are all functions of x^4 only, connected by equations (18). One of the four functions can be arbitrary, and

$$\gamma = \bar{c}x^3 + \bar{b}x^2, \qquad \delta = \bar{g}x^3 + \bar{f}x^1.$$

Case (C 2). We have $\bar{c}^1_{23}\neq 0$, $\bar{c}^2_{13}\neq 0$, and the form of $\bar{\lambda}^i_{s|}$ is given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \alpha & \beta & 1 & 0 \\ \gamma & \delta & 0 & 1 \end{pmatrix}.$$

Case (C 2.1). For this case $\bar{c}^1_{23}$, $\bar{c}^2_{13}$, $\bar{c}^1_{24}$, $\bar{c}^2_{14}\neq 0$, $\bar{c}^1_{34}=\bar{c}^2_{34}=0$, and the g_{ij} is given by

$$\begin{pmatrix} e_1 & 0 & -e_1\alpha & -e_1\gamma \\ 0 & e_2 & -e_2\beta & -e_2\delta \\ -e_1\alpha & -e_2\beta & e_1\alpha^2 + e_2\beta^2 + e_3 & e_1\alpha\gamma + e_2\beta\delta \\ -e_1\gamma & -e_2\delta & e_1\alpha\gamma + e_2\beta\delta & e_1\gamma^2 + e_2\delta^2 + e_4 \end{pmatrix},$$

$\alpha=\bar{a}(x^3, x^4)x^2$, $\beta=-e_1e_2\bar{a}x^1$, $\gamma=\bar{b}(x^3, x^4)x^2$, $\delta=-e_1e_2\bar{b}x^1$, $\bar{a}$ and $\bar{b}$ arbitrary subject to $\partial\bar{b}/\partial x^3=\partial\bar{a}/\partial x^4$.

As stated in I all metric spaces with geodesic Ricci curves for $n=3$ have been obtained. For $n=4$ or greater, this does not seem possible. In this paper we have solved the simplest cases for $n=4$; cases (D) and (E) and the remaining case of (C), in which none of a, b, c, d, f, g are zero, have not been considered.

NORTH CAROLINA STATE COLLEGE

CONCERNING SETS OF POLYNOMIALS ORTHOGONAL SIMULTANEOUSLY ON SEVERAL CIRCLES

G. SZEGÖ

Introduction. Sets of polynomials $\{p_n(z)\}$ simultaneously orthogonal on several curves have been investigated by J. L. Walsh and the author. Recently, the particular case of circles has been treated by G. M. Merriman,* and it is the purpose of this note to give a shorter derivation of his result.

For $r \geqq 0$, $m \geqq 1$ the polynomials

$$1, z, z^2, \cdots, z^{m-1}; \quad z^{n-m}(z^m - r^m), \qquad n = m, m+1, \cdots, \tag{1}$$

are orthogonal on the circles $|z| = R > r$ with the weight function

$$w(z) = |z^m - r^m|^{-2}. \tag{2}$$

From the work of Walsh and the author† we know that if a set of polynomials is orthogonal on two distinct circles, these circles must be concentric, and a very simple relation holds between the corresponding weight functions. According to the results of the author, the case (1), (2) is the only one, save for integral linear transformations, in which a set of polynomials is simultaneously orthogonal on all circles concentric to a given circle and containing it. Merriman shows that this holds true if only the simultaneous orthogonality on two (necessarily concentric) circles is assumed.

Preliminary remarks. In order to prove this pretty result, we shall use the following simple identity satisfied by the polynomials

$$p_n(z) = k_n z^n + \cdots, \qquad k_n > 0; n = 0, 1, 2, \cdots, \tag{3}$$

which constitute an orthonormal set on the unit circle $|z| = 1$ with a preassigned weight function $w(t) = w(e^{i\theta}) = f(\theta)$:

$$\sum_{\nu=0}^{n} \overline{p_\nu(0)} p_\nu(z) = k_n z^n \bar{p}_n(z^{-1}). \tag{4}$$

The reader may find this identity in my earlier paper, *Beiträge zur Theorie der Toeplitzschen Formen*, II (Mathematische Zeitschrift, vol. 9 (1921), pp. 167–190, in particular, p. 174, (33)). For the sake of completeness, however, I include here a very simple direct proof for it.

* See a note of the same title as the present one in this Bulletin, vol. 44 (1938), pp. 57–69. Here also references can be found to the literature on the subject.

† Cf. loc. cit.

We conclude from elementary properties of orthogonal functions that $k_n^{-1}p_n(z)$ gives the minimum value of the integral

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\theta)\,|\,p(z)\,|^2 d\theta, \qquad z = e^{i\theta}, \tag{5}$$

considering the set of all polynomials $p(z)=z^n+\cdots$ of degree n with the highest term z^n. Next, writing

$$z^n\bar{p}(z^{-1}) = u_0p_0(z) + u_1p_1(z) + \cdots + u_np_n(z), \tag{6}$$

we find that this polynomial (6) has the constant term 1 if $p(z)$ has the highest term z^n, so that

$$u_0p_0(0) + u_1p_1(0) + \cdots + u_np_n(0) = 1.$$

Now, according to Cauchy's inequality,

$$\begin{aligned} 1 &= \left|\sum_{\nu=0}^{n} u_\nu p_\nu(0)\right|^2 \leqq \sum_{\nu=0}^{n}|\,u_\nu\,|^2 \sum_{\nu=0}^{n}|\,p_\nu(0)\,|^2 \\ &= \sum_{\nu=0}^{n}|\,p_\nu(0)\,|^2 \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\theta)\,|\,z^n\bar{p}(z^{-1})\,|^2 d\theta \\ &= \sum_{\nu=0}^{n}|\,p_\nu(0)\,|^2 \frac{1}{2\pi}\int_{-\pi}^{+\pi} f(\theta)\,|\,p(z)\,|^2 d\theta, \qquad z = e^{i\theta}, \end{aligned}$$

with the equality sign being taken if and only if

$$u_\nu = \overline{p_\nu(0)}\left\{\sum_{\nu=0}^{n}|\,p_\nu(0)\,|^2\right\}^{-1}, \quad \nu = 0, 1, 2, \cdots, n.$$

Therefore, save for a constant factor, (4) follows. This factor may be determined by comparing the coefficients of z^n.

From (4) we obtain, for $n \geqq 1$,

$$\overline{p_n(0)}p_n(z) = k_nz^n\bar{p}_n(z^{-1}) - k_{n-1}z^{n-1}\bar{p}_{n-1}(z^{-1}). \tag{7}$$

Proof. (i) Assuming that the same system $\{p_n(z)\}$ is orthogonal on a circle $|z| = R > 1$, we determine a sequence of positive numbers $\{\lambda_n\}$ such that $\{\lambda_n^{-1}p_n(Rz)\}$ is an orthonormal set, with a suitable weight function, on $|z| = 1$. Then, for $n \geqq 1$, we may write

$$\lambda_n^{-2}\,\overline{p_n(0)}\,p_n(Rz) = k_nR^n\lambda_n^{-2}z^n\bar{p}_n(Rz^{-1}) - k_{n-1}R^{n-1}\lambda_{n-1}^{-2}z^{n-1}\bar{p}_{n-1}(Rz^{-1}),$$

or

$$\lambda_n^{-2}\,\overline{p_n(0)}\,p_n(R^2z) = k_nR^{2n}\lambda_n^{-2}z^n\bar{p}_n(z^{-1}) - k_{n-1}R^{2n-2}\lambda_{n-1}^{-2}z^{n-1}\bar{p}_{n-1}(z^{-1}). \tag{8}$$

Eliminating $p_{n-1}(z)$ from (7) and (8), we obtain

$$\overline{p_n(0)}\left\{p_n(z) - R^{2-2n}\frac{\lambda_{n-1}^2}{\lambda_n^2}p_n(R^2z)\right\} = k_n\left\{1 - R^2\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\}z^n\bar{p}_n(z^{-1}). \tag{9}$$

(ii) Let $n \geqq 2$, and $p_n(0) \neq 0$. If $1 - R^2\lambda_{n-1}^2/\lambda_n^2 = 0$, (9) shows that $p_n(z) = R^{-2n}p_n(R^2z)$; hence $p_n(z) = \text{const. } z^n$.

If $1 - R^2\lambda_{n-1}^2/\lambda_n^2 \neq 0$, and we set $p_n(z) = \sum_{\nu=0}^{n} c_\nu z_\nu$, we find from (9), for $\nu = 0, 1, 2, \cdots, n$, that

$$\overline{p_n(0)}c_\nu\left\{1 - R^{2-2n+2\nu}\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\} = k_n\left\{1 - R^2\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\}\bar{c}_{n-\nu},$$

and

$$p_n(0)\bar{c}_{n-\nu}\left\{1 - R^{2-2\nu}\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\} = k_n\left\{1 - R^2\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\}c_\nu.$$

Assuming $c_\nu \neq 0$, we have $c_{n-\nu} \neq 0$ and

$$|p_n(0)|^2\left\{1 - R^{2-2n+2\nu}\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\}\left\{1 - R^{2-2\nu}\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\} = k_n^2\left\{1 - R^2\frac{\lambda_{n-1}^2}{\lambda_n^2}\right\}^2, \tag{10}$$

which holds, in particular, for $\nu = 0$ and $\nu = n$. Using this equation, we see that the expression $R^{2-2n+2\nu} + R^{2-2\nu}$ is independent of ν. On the other hand, this expression is decreasing if ν increases from 0 to $n/2$. Consequently, (10) cannot hold unless $\nu = 0$ or $\nu = n$. Thus $p_n(z) = k_n z^n + l_n$. This result includes the case previously considered; in that case we have $l_n = 0$.

(iii) In both cases we obtain from (4) for $n \geqq 2$,

$$\sum_{\nu=0}^{n}\overline{p_\nu(0)}p_\nu(z) = \sum_{\nu=0}^{n-1}\overline{p_\nu(0)}p_\nu(z) + \bar{l}_n(k_nz^n + l_n) = k_n(\bar{l}_nz^n + k_n).$$

Hence $\sum_{\nu=0}^{n-1}\overline{p_\nu(0)}p_\nu(z) = \text{const.}$; that is,

$$p_1(0) = p_2(0) = \cdots = p_{n-1}(0) = 0.$$

As a consequence of this, it follows that $p_m(0) = 0$, $(m > n)$, and

$$k_mz^m\bar{p}_m(z^{-1}) = k_nz^n\bar{p}_n(z^{-1}).$$

Therefore, $k_m = k_n$, and we have the relation

$$p_m(z) = z^{m-n}p_n(z) = z^{m-n}(k_n z^n + l_n).$$

We have assumed, until now, that the sequence $p_2(0), p_3(0), \cdots$ contained a nonzero term. If this is not the case, the last result still holds with $n=1$, as may be seen from (4) in the same way as before.

Now we have, if $m \geqq n$, $m' \geqq n$, $m \neq m'$,

$$\int_{-\pi}^{+\pi} f(\theta)e^{i(m-m')\theta} \left| k_n z^n + l_n \right|^2 d\theta = 0, \qquad z = e^{i\theta}.$$

Whence, except on a set of measure zero, we have

$$f(\theta) = \text{const.} \left| k_n z^n + l_n \right|^{-2}, \qquad z = e^{i\theta}. \tag{11}$$

We conclude the proof with the obvious remark that the polynomials $1, z, z^2, \cdots, z^{n-1}$ are orthogonal on the unit circle $|z| = 1$ with the weight function (11).

STANFORD UNIVERSITY

A FACTORIZATION THEOREM APPLIED TO A TEST FOR PRIMALITY*

D. H. LEHMER

Certain tests for primality based on the converse of Fermat's theorem and its generalizations have been devised and applied by the writer during the past ten years.† Perhaps the most useful test for the investigation of a large number N of no special form may be given as follows:‡

THEOREM 1. *If N divides $a^{N-1}-1$ but is relatively prime to $a^{(N-1)/p}-1$, where p is a prime, then all the possible factors of N are of the form $p^\alpha x+1$, if $N-1$ is divisible by p^α, $(\alpha \geqq 1)$.*

Strictly speaking this is not a test for primality since the theorem merely gives a restriction on the factors of N. If $p^\alpha > N^{1/2}$ then, obviously, N is a prime. If p^α is only fairly large, the theorem gives

* Presented to the Society, February 26, 1938.

† This Bulletin, vol. 33 (1927), pp. 327–340; vol. 34 (1928), pp. 54–56; vol. 35 (1929), pp. 349–350; vol. 38 (1932), pp. 383–384; vol. 39 (1933), pp. 105–108; Annals of Mathematics, (2), vol. 31 (1930), pp. 419–448; Journal of the London Mathematical Society, vol. 10 (1935), pp. 162–165; American Mathematical Monthly, vol. 43 (1936), pp. 347–354.

‡ This Bulletin, vol. 33 (1927), p. 331.

a good restriction on the factors of N and a much better restriction on the values of u in the equation

$$N = u^2 - v^2 = (u + v)(u - v);$$

in fact $2u \equiv N+1 \pmod{p^{2\alpha}}$. If no large divisor p^α of $N-1$ is known, it may be necessary to apply Theorem 1 several times and to combine the several restrictions thus obtained. Finally if no divisor p^α (other than the trivial factor $p^\alpha = 2$) of $N-1$ is known, then Theorem 1 tells us nothing at all.

It thus appears that the problem of proving N a prime is closely allied to the factorization of $N-1$. In general it is impossible to say anything about the factors of $N-1$ beyond the obvious remark that $N-1$ is even. In the interesting case where N is a divisor of a number of the form y^n-1, it is possible to make the factorization of $N-1$ depend upon the factors* of y^k-1, where $k<n$. It is the purpose of this note to indicate how this may be done.

To be more specific, let n be a positive integer, and let $Q_n(x) = x^\phi + \cdots$ be the irreducible polynomial whose roots are the primitive nth roots of unity, so that we have the familiar factorization

$$y^n - 1 = \prod_{\delta \mid n} Q_\delta(y), \tag{1}$$

where δ, as indicated, ranges over all the divisors of n. Then the factorization of numbers of the form y^n-1 depends on the factorization of $N = Q_n(y)$. If we suspect that N is a prime, it may be tested for primality provided something is known of the factors of $N-1$. Before discussing this topic, we need two lemmas.

LEMMA 1.† *If $n = sd$, where s is the product of all distinct prime factors of n, and $d \geqq 1$, then*

$$Q_n(y) = Q_s(y^d). \tag{2}$$

LEMMA 2. *If n is not divisible by the prime q, then*

$$Q_{nq}(y) = Q_n(y^q)/Q_n(y). \tag{3}$$

Both lemmas can be made to follow easily from the familiar‡ Dedekind inversion of (1):

* See Cunningham and Woodall, *Factorization of $y^n \pm 1$*, London, 1925, for extensive tables for $y \leqq 12$. For $y > 12$, see Cunningham, Messenger of Mathematics, vol. 57 (1927), pp. 72–80; see also Kraitchik, *Recherches sur la Théorie des Nombres*, vol. 2, Paris, 1929, pp. 84–159.

† Trudi, Annali di Matematica, (2), vol. 2 (1868–1869), pp. 160–162.

‡ Journal für die reine und angewandte Mathematik, vol. 54 (1857), pp. 25–26.

$$Q_n(y) = \prod_{\delta|n} (y^{n/\delta} - 1)^{\mu(\delta)}, \tag{4}$$

where μ is the Möbius function defined by $\mu(1)=1$, and for $k>1$ by $\mu(k)=(-1)^h$ or zero according as k is a product of h distinct primes or not.

To prove Lemma 1, we note that $\mu(\delta)=0$, except when δ is a divisor of s. Therefore (4) may be written

$$Q_n(y) = \prod_{\delta|s} \{(y^d)^{s/\delta} - 1\}^{\mu(\delta)} = Q_s(y^d).$$

Lemma 2 may be established by noting that the divisors of nq are of the forms δ and $q\delta$, where δ ranges over the divisors of n, so that (4) becomes in this case

$$Q_{nq}(y) = \prod_{\delta|n} (y^{nq/\delta} - 1)^{\mu(\delta)} \prod_{\delta|n} (y^{n/\delta} - 1)^{\mu(q\delta)}. \tag{5}$$

Since q does not divide δ, $\mu(q\delta) = -\mu(\delta)$, and we obtain Lemma 2 at once on comparing (5) with (4).

Returning to the problem of factoring $N-1=Q_n(y)-1$ we find that three cases present themselves according as n has 0, 1, or more than one odd prime factor.

Case 1. If $n=2^\lambda$, we exclude as trivial the cases $n=1$ and 2 in which $Q_1(y)=y-1$, and $Q_2(y)=y+1$. For $n=2^\lambda$, $(\lambda>0)$, (4) gives

$$N = Q_n(y) = (y^n - 1)/(y^{n/2} - 1) = y^{n/2} + 1;$$

so the factorization of $N-1$ is obvious. As a matter of fact, when y is odd N is oddly even, and we are really concerned with $N/2$ and hence with factoring

$$(N/2) - 1 = \tfrac{1}{2}(y^{n/2} - 1) = \tfrac{1}{2}(y - 1)(y + 1)(y^2 + 1) \cdots (y^{n/4} + 1).$$

The decomposition of these binomials into their prime factors may be thought of as known, since their exponents are all considerably less than $n/2$.

Case 2. Assume that $n=2^\lambda q^\beta$, $\lambda \geqq 0$, q an odd prime, $\beta \geqq 1$. In case $\lambda=0$, formula (4) gives

$$N = Q_n(y) = (y^n - 1)/(y^{n/q} - 1).$$

Hence

$$N - 1 = y^{n/q}(y^\phi - 1)/(y^{n/q} - 1),$$

where $\phi=\phi(n)=n-n/q=q^{\beta-1}(q-1)$. In case $\lambda>0$, (4) gives

$$N = Q_n(y) = (y^{n/2} + 1)/(y^{n/2q} + 1).$$

Hence

$$N - 1 = y^{n/2q}(y^{\phi} - 1)/(y^{n/2q} + 1),$$

where, in this case,

$$\phi = \phi(n) = (n/2) - (n/2q) = 2^{\lambda-1}q^{\beta-1}(q - 1).$$

In either case, then, the factorization of $N-1$ can be made to depend on that of binomials of much lower degree. For example, if $n=200$,

$$N - 1 = Q_{200}(y) - 1 = y^{20}(y^{80} - 1)/(y^{20} + 1) = y^{20}(y^{40} + 1)(y^{20} - 1).$$

Case 3. Assume that n has more than one odd prime factor. The final case is much more complicated; it is no longer true that all the prime factors of $N-1$ divide either y or y^m-1 for $m<n$. However, it is possible to account in this way for some of the factors of $N-1$ by the following theorem:

THEOREM 2. *Let $N=Q_n(y)$, and write $n=ds=qtd$, where s is the product of the distinct prime factors of n and q is any prime factor of N; then $N-1$ is divisible by y^d and also by $y^{d(q-1)}-1$, unless t divides $q-1$ in which case $N-1$ is divisible by the integer $(y^{d(q-1)}-1)/Q_t(y^d)$.*

PROOF. It is well known that for every $s>1$, $Q_s(0)=1$. In other words $Q_s(x)-1$ is divisible by x. Hence by Lemma 1

$$N - 1 = Q_n(y) - 1 = Q_s(y^d) - 1 \tag{6}$$

is divisible* by y^d. By Lemma 2, replacing n and y by t and y^d, respectively, we may write, in view of (6),

$$N - 1 = Q_{tq}(y^d) - 1 = \frac{Q_t(y^{dq}) - Q_t(y^d)}{Q_t(y^d)}. \tag{7}$$

The polynomial $Q_t(y^{dq})-Q_t(y^d)$ is divisible by $y^{dq}-y^d$ and hence by $y^{d(q-1)}-1$. In fact, if $f(x)=a_0+a_1x+a_2x^2+\cdots$ is any polynomial, every term of the difference

$$f(u) - f(v) = a_1(u - v) + a_2(u^2 - v^2) + a_3(u^3 - v^3) + \cdots$$

is divisible by $u-v$. Now since (7) is an identity in y^d, we may replace y^d by z and write

$$Q_t(z^q) - Q_t(z) = z(z^{q-1} - 1)P(z), \tag{8}$$

* Incidentally, $N-1$ does not contain a higher power of y than y^d since the penultimate term of $Q_s(x)$ is $-\mu(s)x=\pm x$.

where $P(z)$ is a polynomial with integral coefficients. But the right member of (8) is algebraically divisible by $Q_t(z)$; and since $Q_t(z)$ is irreducible it must divide either $z^{q-1}-1$ or else $P(z)$. By (1) it follows that $z^{q-1}-1$ is divisible by $Q_t(z)$ if and only if t divides $q-1$. Hence the theorem follows at once.

The algebraic factorization of $Q_n(y)-1$ under Case 3 may be given for a few values of n. By Lemma 1, we may confine ourselves to those values of n which are products of distinct primes and since, if k is odd,

$$Q_{2k}(x) = Q_k(-x), \tag{9}$$

we take only odd values of n which have no square factors. For the first five such values we have

$$\begin{aligned}
Q_{15}(y) - 1 &= y(y^4 - 1)(y^3 - y^2 + 1),\\
Q_{21}(y) - 1 &= y(y - 1)(y^3 + 1)(y^7 + y + 1),\\
Q_{33}(y) - 1 &= y(y^{10} - 1)(y^9 - y^8 + y^6 - y^5 + y^3 - y^2 + 1),\\
Q_{35}(y) - 1 &= y(y^2 + 1)(y^6 - 1)(y^{15} - y^{14} - y^{13} + y^{12} + y^{11}\\
&\qquad - y^9 + y^5 - y^2 + 1),\\
Q_{55}(y) - 1 &= y(y - 1)(y^2 + 1)(y^5 + 1)(y^{31} - y^{29} + y^{27} - y^{25}\\
&\qquad + y^{23} + y^{20} - y^{18} + y^{16} - y^{14} + y^{12} + y^{11}\\
&\qquad + y + 1).
\end{aligned} \tag{10}$$

The use of this table is illustrated by the following examples.

Let $N = Q_{60}(11) = 46\ 32945\ 35436\ 00481$. In view of Lemma 1 and (9), we have

$$Q_{60}(11) = Q_{30}(11^2) = Q_{15}(-11^2).$$

Substituting $y = -11^2$ in (10) we find

$$N - 1 = 11^2(11^8 - 1)(11^6 + 11^4 - 1),$$

from which we easily obtain the decomposition into primes

$$N - 1 = 2^5 \cdot 3 \cdot 5 \cdot 11^2 \cdot 61 \cdot 7321 \cdot 1786201.$$

Choosing $p = 1786201$ for the application of Theorem 1 we find that

$$2^{(N-1)/p} - 1 \equiv 25\ 37813\ 44415\ 94865 = A - 1 \pmod{N}$$

and that $A-1$ is prime to N. Furthermore we find that

$$A^p = 2^{N-1} \equiv 1 \pmod{N}.$$

Hence by Theorem 1 the factors of N are of the form $1786201x+1$. Combining* this with $60x+1$ we get a restriction on the factors of N of the form $107172060x+1$. Only two numbers of this form exist below $N^{1/2}$, namely 107172061 and 214344121. Since neither of these is a factor of N, it follows that N is a prime.

As a second example consider

$$N = Q_{240}(2) = 18518\ 80056\ 39241\ 07521.$$

By Lemma 1 and (9) we have

$$N = Q_{240}(2) = Q_{30}(2^8) = Q_{15}(-2^8).$$

Hence by (10)

$$N-1 = 2^8(2^{32}-1)(2^{24}+2^{16}-1) = 2^8\cdot 3\cdot 5\cdot 17\cdot 257\cdot 65537\cdot 16842751,$$

the last factor being easily identified as a prime.† Armed with this complete factorization of $N-1$ we are now in a position to apply Theorem 1 to the number N, which has been investigated and declared prime by M. Kraitchik‡ in 1929 by a plausible though non-rigorous method.§ It turns out however that the first part of the hypothesis of Theorem 1 is not satisfied, that is, N does not divide $11^{N-1}-1$. Hence, by Fermat's theorem, N is composite. To settle this question we applied a new type of machine for combining linear congruences to the problem of factoring N with the result that

$$N = 394783681\cdot 46908728641.$$

LEHIGH UNIVERSITY

---

* Every factor of $Q_n(y)$ not dividing n is of the form $nx+1$.

† By the method given in the American Mathematical Monthly, vol. 43 (1936), pp. 347–354.

‡ *Recherches sur la Theorie des Nombres*, vol. 2, Paris, 1929, pp. 12–17.

§ This Bulletin, vol. 36 (1930), pp. 847–850.

REPRESENTATION OF HOMEOMORPHISMS IN HILBERT SPACE

MORRIS KLINE

1. **Introduction.** Given a normal, topological space S and a continuous transformation f of S into itself, it is possible to imbed S (homeomorphically) in a cartesian space (generally of uncountable dimension) so that the transformation f can be represented as a simple linear transformation of the form

$$y_\alpha = x_\beta, \tag{1}$$

where y_α is the αth coordinate of the transformed point and x_β the βth coordinate of the original point, and α runs through all the coordinate indices. However, the methods of imbedding S whereby such a representation of f becomes possible generally do not permit us to simplify the dimension or structure of the cartesian space when S is more specialized.

Tychonoff has shown* that every normal† topological space S with a neighborhood system of power less than or equal to τ can be imbedded in (a bicompact part of) a cartesian space R_τ in which each point has τ real numbers as coordinates. In particular, if $\tau = \aleph_0$, S can be imbedded in the fundamental parallelopiped of Hilbert space.‡ Now it is possible to refine Tychonoff's method slightly so that a homeomorphism g of S onto itself can be represented by equations of the form (1) acting in R_τ. And when S has a countable neighborhood system, the equations (1) will act in the fundamental parallelopiped of Hilbert space. Thus, to represent a homeomorphism of these more restricted spaces by (1) we can confine ourselves to the fundamental parallelopiped of Hilbert space.

2. **Tychonoff's method.** Tychonoff's method of imbedding a normal topological space S with neighborhood system of power less than or equal to τ in a cartesian space R_τ is as follows.§ We may take as our neighborhood system a basis B for the open sets of S with power

* A. Tychonoff, *Topologische Erweiterung von Räumen*, Mathematische Annalen, vol. 102 (1929), pp. 544–561.

† Complete regularity is sufficient.

‡ By the fundamental parallelopiped of Hilbert space we mean the set of all points $x = (x_1, x_2, \cdots, x_n, \cdots)$ such that $x_n \leqq 1/n$ and such that the distance between two points is given by $[\sum_1^\infty (x_i - y_i)^2]^{1/2}$.

§ Loc. cit., pp. 551–552.

τ.* A pair of sets (U, V) of the basis is called canonical if $U \supset \bar{V}$ and if there exists at least one continuous function $f(x)$ on S such that $f=0$ on $\bar{V}$, $f=1$ on $S-U$, and for every x of S, $0 \leqq f(x) \leqq 1$. There are exactly τ such canonical pairs, and for each one we select one $f(x)$ satisfying these conditions. To these functions we assign distinguishing indices $1, 2, \cdots, \alpha, \cdots$. Then if x is any point of S, we let it correspond to that point x of R_τ which has the coordinates $(f_1(x), f_2(x), \cdots, f_\alpha(x), \cdots)$. We make the following choice of neighborhoods in R_τ. For any ϵ and any choice of a finite number of indices $\alpha_1, \alpha_2, \cdots, \alpha_m$, we understand by a neighborhood of a point x of R_τ all points of R_τ whose α_ith coordinate $(i=1, 2, \cdots, m)$ differs from the α_ith coordinate of x by less than ϵ. With this choice of neighborhoods in R_τ the mapping, just described, of S onto a subset of R_τ is shown by Tychonoff to be a homeomorphism. If S has a countable basis, R_τ is $R_{\aleph_0}$. In this case we make the additional mapping into the fundamental parallelopiped of Hilbert space by means of the homeomorphism

$$x = (x_1, x_2, \cdots, x_n, \cdots), \qquad x' = (x_1, x_2/2, \cdots, x_n/n, \cdots).$$

We note that for the purpose of mapping S into R_τ we can associate to a canonical pair (U, V) any continuous function on S which satisfies the conditions listed above.

3. **Refinement of Tychonoff's method.** Now let S be a normal, topological space, and let g be a homeomorphism of S onto itself.

By an irreducible basis of a space S we mean a basis for the open sets from which no set can be omitted without the loss of the basis property. Any basis gives rise to at least one irreducible basis by omitting as many superfluous sets as necessary.

If the system $\{U\}$ is an irreducible basis for S, then $g(U)$ belongs to this basis. For if not, then $g(U) = U'$ is a sum of sets V_i' each of which belongs to the basis. For each i, $g^{-1}(V_i') = V_i$ is a proper subset of U and is an open set. Moreover, each of these V_i is either a set of the basis or a sum of such sets. Then $U = \sum_i V_i$ is an exact sum of sets of the basis, and the basis is not irreducible.

By the same argument it follows that if U is a set of an irreducible basis, then $g^{-1}(U)$, that is, the pre-image under g of U, is a set of this basis.

If (U, V) is a canonical pair chosen from an irreducible basis, then the pre-images under g of U and V, say U_1 and V_1, form a canonical pair. For we can and do agree to take as a continuous function satis-

* P. Alexandroff and H. Hopf, *Topologie*, vol. 1, p. 42, Theorem VIII.

fying the conditions mentioned in §2 the function which has that value on any point x of S which the function corresponding to the canonical pair (U, V) has on $g(x)$.

Now we regard S as imbedded in the Cartesian space R_τ by the method indicated in §2. It is a simple consequence of the definitions of a homeomorphism and of a real continuous function on a space* that the homeomorphism g of S onto itself can be represented in terms of the coordinates of the Cartesian space in which S is imbedded by means of the equations

$$y_\alpha = g_\alpha(x_1, x_2, \cdots, x_\gamma, \cdots), \tag{2}$$

where α and γ run through all the indices of coordinates in R_τ and each g_α is some real continuous function on the points of S in R_τ.

Because g is a homeomorphism, y_α must take on exactly the set of values of the αth coordinates of points of S. This set of values was determined in the imbedding of S in R_τ by a function f_α corresponding to a canonical pair (U, V). That is, g_α takes on the set of values on S that f_α does. If we take the pre-images under g of U and V, we get a canonical pair (U_1, V_1) such that g_α acts with respect to (U_1, V_1) as f_α acts with respect to (U, V). Because all canonical pairs are used in the imbedding process, and because we have agreed that canonical pairs related as (U, V) and (U_1, V_1) are should have continuous functions attached to them such that the function corresponding to (U_1, V_1), say f_β, has the value on any point x of S that f_α has on $g(x)$, it must be that g_α is f_β. Since f_β determines the x_βth coordinate in R_τ, we may write (2) as

$$y_\alpha = x_\beta, \tag{3}$$

where α runs through all the coordinate indices but the same x_β's may happen to correspond to different y_α's.

If S has a countable basis, the equations (3) are countable in number and represent a transformation in $R_{\aleph_0}$. Then the transformation indicated in §2 which carries S from a subset of $R_{\aleph_0}$ to a subset of the fundamental parallelopiped of Hilbert space does not affect the form of equations (3); hence these equations represent the homeomorphism g in this parallelopiped of Hilbert space.

INSTITUTE FOR ADVANCED STUDY

---

* Cf. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, 1934, p. 25.

A CREMONA INVOLUTION IN S_3 WITHOUT A SURFACE OF INVARIANT POINTS*

VIRGIL SNYDER AND EVELYN CARROLL-RUSK

1. **Definition.** A Cremona involution in S_3, which is related to the involutions defined in a previous paper,† is of particular interest as it has no surface of invariant points.

Let

$$|H| \equiv H_1(x) - \lambda H_2(x) = 0 \tag{1}$$

be a pencil of quadric surfaces in S_3 and

$$\sum a_i x_i \equiv (ax) = 0, \qquad (bx) = 0 \tag{2}$$

a system of lines g in which each a_i is a polynomial of order m_1 in λ and each b_i a polynomial of order m_2 in λ.

The line g', polar conjugate of g as to H, is defined by

$$\begin{aligned} r_{32}\frac{\partial H}{\partial x_1} + r_{13}\frac{\partial H}{\partial x_2} + r_{21}\frac{\partial H}{\partial x_3} &= 0, \\ r_{42}\frac{\partial H}{\partial x_1} + r_{14}\frac{\partial H}{\partial x_2} + r_{21}\frac{\partial H}{\partial x_4} &= 0, \qquad r_{ik} = a_i b_k - a_k b_i. \end{aligned} \tag{3}$$

If y is any point in space, the H of the pencil through y is determined by $\lambda = H_1(y)/H_2(y)$. Through the point y passes one transversal line t of g and g'. Its equations are $(ax)(by) - (bx)(ay) = 0 = \sum r_{ik} p_{ik} = 0$, where

$$p_{ik} = x_i y_k - x_k y_i, \tag{4}$$

and the corresponding equation from (3), which, after reduction by means of the quadratic identities in the r_{ik}, and so on, may be written in the form

$$\sum r_{ik} H_{lm} = 0, \qquad i, k, l, m \text{ all different}, \tag{5}$$

in which

$$H_{lm} \equiv \frac{\partial H}{\partial x_l}\frac{\partial H}{\partial y_m} - \frac{\partial H}{\partial x_m}\frac{\partial H}{\partial y_l}.$$

---

* Presented to the Society, April 16, 1938.

† Snyder and Carroll-Rusk, *A series of involutorial transformations in S_n belonging multiply to a non-linear line complex*, American Journal of Mathematics, vol. 59 (1937), pp. 775–782.

The equation (5) is linear in x and in y and contains λ to degree m_1+m_2+2; it can be represented, for example, by $\sum A_i x_i = 0$.

The coordinates f_i of the point F in which t meets g are given by the determinants of the third order in the matrix

$$\begin{Vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ A_1 & A_2 & A_3 & A_4 \end{Vmatrix}.$$

Thus, for example,

$$f_1 = r_{23}A_4 + r_{34}A_2 + r_{42}A_3. \tag{6}$$

The coordinates f_i are linear in y and are of order $2(m_1+m_2+1)$ in λ. For any given λ the equations (6) define a harmonic homology.

The line t meets $H(y)$ again in the point y' defined by $\rho y_i' = \sigma f_i + \tau y_i$, in which $\sigma = H(y, f)$, $\tau = -H(f)$. The transformation $y \sim y'$ is a Cremona involutorial transformation I of S_3 into itself.

The surface $\sigma = 0$ is the locus of invariant points of I. Since from the definition of I there is no such surface, σ must divide out of the transformation and hence be a factor of τ.

2. **Fundamental lines of the second kind.** For a given point F, the surface $\sigma = H(f, y) = 0$ is the polar plane of F as to its associated quadric. When F is on H, σ is the tangent plane to H at F. Let g meet H at z and w. When F is at z, $H(z, y) = 0$ is the tangent plane to H at z, which contains g'. If g' meets H at z' and w', zz' lies on H and meets g and g'; hence it is a fundamental line of the second kind, every point of it having the whole line for conjugate; similarly for zw', wz', ww'. The locus of these lines is the surface $\sigma = 0$. Its equation is linear in y_i and in f_i, which contains y linearly and λ to multiplicity $2(m_1+m_2+1)$. Moreover, $H(f, y)$ involves λ explicitly to multiplicity 1; hence the surface is of order $4(m_1+m_2+2)$ and contains the base C_4 of $|H|$ to multiplicity $2(m_1+m_2)+3$.

Since τ is $H(f)$, it is quadratic in y and contains λ to multiplicity $4(m_1+m_2)+5$.

After removing the factor σ from τ, the quotient is of order $2(m_1+m_2+1)$ in λ and does not contain y. The reduced form of the equations of I is then $\rho y_i' = f_i + h y_i$, $h = \tau/\sigma$. The transformation is of order $4(m_1+m_2)+5$ and contains the base C_4 to multiplicity $2(m_1+m_2+1)$. Since each quadric of the pencil is transformed into itself by I, it follows that the principal surface M, conjugate to C_4, is of order $8(m_1+m_2+1)$ and contains C_4 to multiplicity $4(m_1+m_2)+3$.

3. **Isolated fundamental points.** For certain values of λ the line g touches its associated quadric H. They are given by the tact-invariant of H, obtained by bordering the discriminant of H by the two sets of elements a_i, b_i and equating to zero. It is of order $2(m_1+m_2+1)$ in λ. This tact-invariant is $h=\tau/\sigma$. The image of each point of contact P_i is the quadric H of the pencil which contains it. These points are therefore double on the web of surfaces, conjugates to the planes of space. They and C_4 are the only base elements of the web and completely define it.

A point on C_4 has for its conjugate a rational curve of order $2(m_1+m_2+1)$, expressed parametrically by $\rho x_i=f_i(\lambda)$, wherein y is replaced by a point ξ of C_4 and λ is the independent parameter. Then if ξ is eliminated by means of $H_1(\xi)=0$, $H_2(\xi)=0$, the equation $M=0$ results. The table of characteristics can now be written: If $m_1+m_2=m$, then

$$
\begin{aligned}
S_1 &\sim S_{4m+5}\colon\ C_4^{2(m+1)}2(m+1)P_i^2,\\
C_4 &\sim M_{8(m+1)}\colon\ C_4^{4m+3}2(m+1)P_i,\\
P_i &\sim H_{2,i}\colon\ C_4P_i,\\
J &\equiv M2(m+1)H_{2,i}^2.
\end{aligned}
$$

The locus of the points of intersection (g, H) lies on the ruled surface R, locus of g, on the locus $A=0$, generated by $(ax)=0$ and $H_1=\lambda H_2$, and also on $B=0$, generated by $(bx)=0$ and $H_1=\lambda H_2$. It is of order $2m+1$. Any point of the curve, which is always hyper-elliptic, is invariant under I. Similarly, the locus (g', H), which may be of different order, may be determined.

The congruence of bisecants of C_4 is transformed into itself.

4. **Case of m_1 or $m_2=0$.** If the line g lies in a plane, then one of its equations does not contain λ and the plane is invariant under I. The plane meets each g' in a point G', pole of g as to the associated conic; hence the plane involution I' can be defined by means of a pencil of conics and a projective rational curve. If the locus of G' is of order n, the table of characteristics is

$$
\begin{aligned}
C_1 &\sim C_{4n+3}\colon\ 4B^{2n+1}(2n+1)P_i^2,\\
B &\sim \beta_{2n+1}\colon\ B^{n+1}3B_k^n,\\
P_i &\sim C_2\colon\ 4BP_i,\\
\sigma_{2n+3} &\equiv \sigma\colon\ 4B^{n+1}(2n+1)P_i,\\
J &\equiv 4\beta_{2n+1}(2n+1)C_2.
\end{aligned}
$$

As is thus seen, this is a semi-symmetric involution of Ruffini.*

By considering the various types of pencils of conics and choosing points on the locus of G' in all possible relations as to the composite conics of the pencil, all the non-monoidal semi-symmetric series of Ruffini involutions can be obtained. Most of the theorems already appearing in the literature follow immediately from this point of view.

5. **Map of** I. For many problems it is desirable to have a map of an involution, such that a point x' of the map represents a point y and its conjugate under I. When x' describes a linear space S', I is said to be rational.

There are three systems of invariant surfaces. If the equations of I are $\rho y_i' = \phi_i(y)$, these may be written:

$$\sum a_i y_i \phi_i = 0, \quad \infty^3 \text{ system},$$

$$\sum a_i (y_i \phi_k - y_k \phi_i) = 0, \quad \infty^4 \text{ system},$$

$$\sum a_i (y_i \phi_k + y_k \phi_i) = 0, \quad \infty^5 \text{ system}.$$

Each system defines a map of the given involution. If it is assumed that $H_1(y) = x_1'$, $H_2(y) = x_2'$, by this scheme each quadric of the pencil is mapped upon a plane of a pencil; since by I each quadric of the pencil remains invariant, the pencil of planes furnishes an adequate map. If the space (y) is thought of as superposed on (x') with the same coordinate system and the same system of directrices g, g', the map has no surface of branch points since I contains no surface of invariant points.

The only fundamental points of the map are those of the axis of the pencil planes. The equations are

$$\rho x_i' = (H_2 y_1 - H_1 y_2) f_i - (H_2 f_1 - H_1 f_2) y_i.$$

The quadrics associated with contact on P_i are projected upon their image plane stereographically, the general quadric by a harmonic axial homology.

CORNELL UNIVERSITY AND
WELLS COLLEGE

---

* *Sulla risoluzione delle due equazioni di condizione delle trasformazioni Cremoniane delle figure piane*, Memorie dell'Accademia dell' Istituto, Bologna, (3), vol. 8 (1877), pp. 457–525. D. Montesano, *Le corrispondenze birazionali piane semisimmetriche*, Rendiconti dell'Accademia, Naples, (3), vol. 24 (1918), pp. 31–78.

A DUALITY FOR CERTAIN DIFFERENCE EQUATIONS

E. T. BELL

1. **Umbral factorials and Appell polynomials.** The difference equations considered include as special cases those satisfied by the generalized Bernoulli and Euler polynomials,* also many of interest in the theory of numbers; the duality was primarily devised for the investigation of divisibility properties of certain sequences of rational numbers obtained from generalizations of the polynomials mentioned. The connection with arithmetic is made through Fermat's theorem and Lagrange's identical congruence, and is developed elsewhere.† In constructing the duality, an inconsistency in the usual statement of symbolic equations that has been overlooked by writers on the symbolic methods is uncovered (§1, end). This inconsistency is trivial for the customary uses of the symbolic method, but until it is rectified it is impossible to proceed to the new applications of the method made here.

The duality is most simply displayed in Blissard's symbolism (suitably amplified), in which α is the umbra of the sequence α_n, $(n=0, 1, \cdots)$, or of the vector $(\alpha_0, \alpha_1, \cdots)$, and the symbolic power α^n denotes α_n. Small Greek letters without suffixes, α, $\alpha^{(n)}$, β, $\beta^{(n)}$, $\cdots$, and $S^{(n)}$, will be used exclusively for umbrae, small Latin letters, with or without suffixes, a, a_i, $\cdots$, x, x_i, t, $\cdots$ for ordinaries (real or complex numbers). Small Greek letters with suffixes, as α_n, $\alpha_r^{(n)}$, $\cdots$, denote ordinaries. By definition, $x\alpha$ is the umbra of $x^n\alpha_n$, $(n=0, 1, \cdots)$. Umbral equality, $\alpha=\beta$, signifies $\alpha^n=\beta^n$, that is, $\alpha_n=\beta_n$, $(n=0, 1, \cdots)$. If $\alpha\neq\beta$, α, β are said to be distinct.

Provided the series converges for some $|t|>0$,

$$e^{x\alpha t} \equiv \sum_{n=0}^{\infty} x^n\alpha^n \frac{t^n}{n!} \equiv \sum x^n\alpha_n \frac{t^n}{n!}.$$

Hence, if $\alpha, \beta, \cdots, \gamma$ are all distinct, and $xy\cdots z\neq 0$,

$$e^{x\alpha t}e^{y\beta t}\cdots e^{z\gamma t} = e^{(x\alpha+y\beta+\cdots+z\gamma)t},$$

convergence of the umbral exponentials on the left being assumed, where $(x\alpha+y\beta+\cdots+z\gamma)^s$, $(s\geqq 0)$, is to be replaced after expansion

* N. E. Nörlund, *Differenzenrechnung*, chap. 6.

† In a paper on generalized Stirling transforms of sequences to appear in the American Journal of Mathematics, 1939.

of the right by its equivalent as given by the umbral multinomial theorem,

$$(x\alpha + y\beta + \cdots + z\gamma)^0 = \alpha_0\beta_0 \cdots \gamma_0,$$

$$(x\alpha + y\beta + \cdots + z\gamma)^s = \sum M_{a,b,\cdots,c} x^a y^b \cdots z^c \alpha_a \beta_b \cdots \gamma_c, \quad s > 0,$$

in which the M's are multinomial coefficients from the ordinary multinomial theorem

$$(x + y + \cdots + z)^s = \sum M_{a,b,\cdots,c} x^a y^b \cdots z^c.$$

The exponents 0, 1 of umbrae are treated precisely as any other non-negative exponents, in distinction to the like in a field, where $x^0 = 1$, $x^1 = x$. Thus $\alpha^0 = \alpha_0$, $\alpha^1 = \alpha_1$.

The umbrae η, ω, $\delta^{(s)}$, $S^{(s)}$ are special:

(1) $\eta_0 = 1$, $\eta_n = 0$, $(n > 0)$; $\omega_n = 0$, $(n \geqq 0)$;

(2) $\delta^{(s)}$ is the umbra of $(\delta_0^{(s)}, \delta_1^{(s)}, \cdots)$, where $\delta_r^{(s)}$ is a Kronecker delta, $\delta_s^{(s)} = 1$, $\delta_r^{(s)} = 0$, $(r \neq s)$.

The (ordinary) factorial $(t)_n$ is defined by

$$(t)_0 = 1, \qquad (t)_n \equiv \prod_{s=0}^{n-1} (t - s) = \sum_{s=1}^{n} S_n^{(s)} t^s, \qquad n > 0,$$

where the $S_n^{(s)}$ are the Stirling numbers of the first kind, $S_0^{(0)} = 1$, $S_n^{(s)} = 0$, $(s > n)$. If $|t| < 1$, there is the known expansion

$$[\log (1 + t)]^s = s! \exp [S^{(s)} t], \tag{3}$$

in which the nth power $(n \geqq 0)$ of $S^{(s)}$ is replaced after expansion by $S_n^{(s)}$.

For any umbra ξ the (umbral) factorial $(\xi)_n$ is defined by

$$(\xi)_0 \equiv \xi_0, \qquad (\xi)_n \equiv \prod_{s=0}^{n-1} (\xi - s) = \sum_{s=1}^{n} S_n^{(s)} \xi_s, \qquad n > 0. \tag{4}$$

The umbra of $(\xi)_n$, $(n = 0, 1, \cdots)$, is written (ξ). If for some $|t| > 0$ the series converge,

$$(1 + t)^{\xi} = e^{(\xi)t}, \tag{5}$$

where the left is expanded formally by the binomial theorem (as if ξ were ordinary), the right by the exponential theorem, and ξ^n, $(\xi)^n$ are replaced after expansion by ξ_n, $(\xi)_n$, $(n \geqq 0)$.

The (ordinary) Appell polynomial $A_n(t, \alpha)$ in t with the base α is $A_0(t, \alpha) \equiv \alpha_0$,

$$A_n(t, \alpha) \equiv (t + \alpha)^n = \sum_{s=0}^{n} (n, s) \alpha_s t^{n-s}, \qquad n > 0,$$

where (n, s) is the binomial coefficient $n!/s!(n-s)!$. The (umbral) Appell polynomial $A_n(\xi, \alpha)$ in ξ with the base α is $A_0(\xi, \alpha) \equiv \alpha_0\xi_0$,

$$A_n(\xi, \alpha) \equiv (\xi + \alpha)^n = \sum_{s=0}^{n} (n, s)\alpha_s\xi_{n-s}, \qquad n > 0; \tag{6}$$

whence $A(\xi, \alpha) = A(\alpha, \xi)$. If τ is the umbra of $(1, t, t^2, \cdots)$, then $A(\tau, \alpha) = A(t, \alpha)$. Thus the umbral Appell polynomials generalize the ordinary ones.

Let ξ be distinct from α, β. Umbral derivation, D_ξ, with respect to ξ is defined by

$$D_\xi\alpha_m\xi_n = \alpha_m D_\xi\xi_n = n\alpha_m\xi_{n-1}, \qquad n > 0,$$
$$D_\xi\alpha_m\xi_0 = 0, \qquad D_\xi(\alpha_m\xi_n + \beta_r\xi_s) = D_\xi\alpha_m\xi_n + D_\xi\beta_r\xi_s.$$

Hence, generalizing the characteristic property of Appell polynomials, we have $D_\xi A_0(\xi, \alpha) = 0$,

$$D_\xi A_n(\xi, \alpha) = nA_{n-1}(\xi, \alpha), \qquad n > 0. \tag{7}$$

We have also, by (1),

$$A(\xi, \eta) = \xi, \qquad A(\xi, \omega) = \omega. \tag{8}$$

In the definition of (ξ) as the umbra of the sequence $(\xi)_n$, $(n = 0, 1, \cdots)$, of umbral factorials constructed from ξ, ξ is arbitrary. Hence the umbra of the sequence $(A(\xi, \alpha))_n$, $(n = 0, 1, \cdots)$, of factorials constructed from $A(\xi, \alpha)$ is written $(A(\xi, \alpha))$. In the definition (4), ξ is any umbra. Hence, replacing ξ in (4) by $A(\xi, \alpha)$, we have $(A(\xi, \alpha))_0 = \xi_0\alpha_0$,

$$(A(\xi, \alpha))_n = \sum_{s=1}^{n} S_n^{(s)} A_s(\xi, \alpha), \qquad n > 0. \tag{9}$$

It may be shown from the properties of the Stirling numbers that

$$(A(\xi, \alpha))_n = [(\xi) + (\alpha)]^n, \qquad n = 0, 1, \cdots;$$

that is, by the definition of umbral equality, we have the second of the next two equations,

$$A(\xi, \alpha) = \xi + \alpha, \qquad (A(\xi, \alpha)) = (\xi) + (\alpha), \tag{10}$$

the first of which is equivalent to (6). Thus the first of (10) implies the second. It can be shown in the same way that the second implies the first.

An implied restriction on ordinary constants occurring in equations containing umbrae appears to have been overlooked by writers on

the umbral calculus, as its necessity for consistency can be ignored until the umbral equations are dualized as in the next section. The point is sufficiently evident from an example. Let E be the umbra of the Euler numbers. Then one of the usual umbral equations for the E_n is customarily written

$$(E+2)^n + E^n = 2, \qquad n = 0, 1, \cdots .$$

The terms on the left are umbrae, the constant 2 on the right is an ordinary, and it is meaningless to equate an umbra to an ordinary, although the (meaningless) equation gives correct results in the classic applications of the symbolic method. The equation is derived from

$$\operatorname{sech} t = 2e^t/(e^{2t}+1) \equiv e^{Et},$$

which may be taken as a definition of E. Proceeding as usual we get

$$e^{(E+2)t} + e^{Et} = 2e^t$$

and hence, on equating coefficients of t,

$$(E+2)^n + E^n = 2 \cdot 1^n,$$

the correct form, in which 2 appears as $2 \cdot 1^n$. The 1 in 1^n is the umbra of $(1^0, 1^1, 1^2, \cdots)$, namely of $(1, 1, 1, \cdots)$; and it is clear why the incorrect equation gives correct results. Generally, if an equation involving umbrae contains also an ordinary term c, this term must be interpreted as the umbra of $(c, c1, c1^2, \cdots)$. Similarly, in $\xi + c$, c is the umbra of $(1, c, c^2, \cdots)$.

2. **Duality.** Two relations, each of which implies the other, will be called duals. The dual difference equations considered are obtained as follows. Let $\alpha^{(r)}, \beta^{(s)}, \cdots$ be all the umbrae other than the special umbrae $\delta^{(u)}, S^{(v)}$, or umbrae of the type noted at the end of the preceding section, occurring in the first equation of a pair of duals. Let the first equation be between nth powers of umbrae, valid for $n = 0, 1, \cdots$. Then passage from the first equation to the second is made by the substitution

$$\alpha^{(r)} \to (\alpha^{(r)}), \quad \beta^{(s)} \to (\beta^{(s)}), \cdots ,$$

and, if one or more of $\delta^{(u)}, S^{(v)}, ca^n$ occurs, by the corresponding substitutions from among

$$\delta^{(u)} \to S^{(u)}, \qquad S^{(v)} \to \delta^{(v)}, \qquad ca^n \to c(a)_n .$$

Passage from the second equation to the first is made by the inverse substitutions

$$(\alpha^{(r)}) \to \alpha^{(r)}, \qquad (\beta^{(s)}) \to \beta^{(s)}, \cdots,$$
$$S^{(u)} \to \delta^{(u)}, \qquad \delta^{(v)} = S^{(v)}, \qquad c(a)_n \to ca^n.$$

For example, E being as in §1, end, we have the duals

$$(E+2)^n + E^n = 2\cdot 1^n,$$
$$[(E)+(2)]^n + (E)^n = 2(1)_n, \qquad n \geqq 0.$$

Now $(c)_0=1$, $(c)_n=c(c-1)\cdots(c-n+1)$, $(n>0)$. Hence $(2)_0=1$, $(2)_1=(2)_2=2$; $(2)_n=0$, $(n>2)$; $(1)_0=1$, $(1)_n=0$, $(n>1)$; and since

$$[(E)+(2)]^n = \sum_{s=0}^{n} (n, s)(E)_{n-s}(2)_s,$$

the above duals in expanded form are

$$E_0 = 1, \qquad E_n + \sum_{s=1}^{n} 2^{s-1}(n, s)E_{n-s} = 1, \qquad n > 0;$$

$$(E)_0 = 1, \quad 2(E)_{n+2} + 2(n+2)(E)_{n+1} + (n+1)(n+2)(E)_n = 0, \quad n \geqq 0.$$

The solution of the last is $(E)_n = n!a_n$, $(n \geqq 0)$, where

$$a_{4n}=(-1)^n 2^{-2n}, \quad a_{4n+1} = 0, \quad a_{4n+2}=(-1)^{n+1}2^{-2n-1}, \quad a_{4n+3}=(-1)^n 2^{-2n-1}.$$

With the definitions $0^0=(0)_0=1$, we have also the duals

$$(E+1)^n + (E-1)^n = 2\cdot 0^n,$$
$$[(E)+(1)]^n + [(E)+(-1)]^n = 2(0)_n, \qquad n \geqq 0.$$

3. **Duals.** We shall give one pair, (15), (15′), of duals of a very general character. Proofs are deferred for the moment.

The η, ω, $\delta^{(n)}$, $S^{(n)}$ have the special meanings assigned in §1; h, i, j, k are nonnegative integers ranging, respectively, from 0 to l, 0 to m, 0 to n, 0 to r; L_h, M_i, N_j, R_k are any given ordinary constants, $\lambda^{(h)}$, $\mu^{(i)}$, $\nu^{(j)}$, $\rho^{(k)}$ any given umbrae such that

$$N_0\nu_0^{(0)} + \sum_{k=0}^{r} R_k\rho_0^{(k)} \neq 0. \tag{11}$$

It follows that the θ_n, $(n=0, 1, \cdots)$, given by

$$\sum_{j=0}^{n} j!N_j[\theta + A(\nu^{(j)}, \delta^{(j)})]^u + \sum_{k=0}^{r} R_k[\theta + \rho^{(k)}]^u$$
$$= \sum_{h=0}^{l} h!L_h A_u(\lambda^{(h)}, \delta^{(h)}) + \sum_{i=0}^{m} M_i\mu_u^{(i)}, \tag{12}$$

for $u=0, 1, \cdots$, are determinate when, by convention, the binomial coefficient $(0, 0)=1$, $(0, s)=0$, $(s>0)$.

The dual (12′) of (12) is the equation satisfied by (θ). It is written down from (12) by the substitution

$$\begin{gathered}\theta, \nu^{(j)}, \rho^{(k)}, \lambda^{(h)}, \mu_u^{(i)} \to (\theta), (\nu^{(j)}), (\rho^{(k)}), (\lambda^{(h)}), (\mu^{(i)})_u; \\ \delta^{(j)}, \delta^{(h)} \to S^{(j)}, S^{(h)}.\end{gathered} \tag{13}$$

An alternative form (12″) of (12′) is given from (12′), by the second equation of (10), by the substitution

$$\begin{gathered}\theta, \mu^{(i)}, \rho^{(k)} \to (\theta), (\mu^{(i)}), (\rho^{(k)}); \\ A(\nu^{(j)}, \delta^{(j)}), A(\lambda^{(h)}, \delta^{(h)}) \to (A(\nu^{(j)}, \delta^{(j)})), (A(\lambda^{(h)}, \delta^{(h)})).\end{gathered} \tag{14}$$

We now generalize (12), (12′), (12″) by constructing the equations satisfied by the umbral Appell polynomials $A_n(\xi, \theta)$, $(n=0, 1, \cdots)$, in ξ with the base θ, and by the factorials $(A(\xi, \theta))_n$ constructed from the $A_n(\xi, \theta)$. The equation for $A(\xi, \theta)$ can be written down from (12); that for $(A(\xi, \theta))$ is then obtained immediately by dualizing. We find

$$\begin{aligned}\sum_{j=0}^{n} j!N_j[A(\xi, \theta) + A(\nu^{(j)}, \delta^{(j)})]^u + \sum_{k=0}^{r} R_k[A(\xi, \theta) + \rho^{(k)}]^u \\ = \sum_{h=0}^{l} l!L_h[\xi + A(\lambda^{(h)}, \delta^{(h)})]^u + \sum_{i=0}^{m} M_i[\xi + \mu^{(i)}]^u, \\ u = 0, 1, \cdots.\end{aligned} \tag{15}$$

This follows from (12) by the first equation of (10) on replacing θ in (12) by $\xi+\theta$, expanding the result in powers of ξ, and using (12) with $u=0, 1, \cdots$ to identify coefficients of ξ_u in (15). As already noted, $A_n(\xi, \theta)$ is the ordinary Appell polynomial $A_n(x, \theta)$ in x with the base θ if ξ is its special case, the umbra of $(1, x, x^2, \cdots)$. To obtain (12) from (15) we take $\xi \equiv \eta$ and refer to the first of (8). The dual (15′) of (15) may be stated in each of several equivalent forms. One is obtained by the substitution

$$A(\xi, \theta), A(\nu^{(j)}, \delta^{(j)}), A(\lambda^{(h)}, \delta^{(h)}) \to (A(\xi, \theta)), (A(\nu^{(j)}, \delta^{(j)})), (A(\lambda^{(h)}, \delta^{(h)}))$$

in (15), with $\rho^{(k)} \to (\rho^{(k)})$ on the left and $\xi, \mu^{(i)} \to (\xi), (\mu^{(i)})$ on the right. We prove the foregoing results first under a convergence restriction which will be removed presently. Assume for a moment that all the expressions

$$\exp \lambda^{(h)}t, \qquad \exp \mu^{(i)}t, \qquad \exp \nu^{(j)}t, \qquad \exp \rho^{(k)}t$$

when written out as ordinary MacLaurin series are absolutely con-

vergent for some $|t| > 0$. By (11), the like is true for $Q(t) \equiv F(t)/G(t)$, where

$$
\begin{aligned}
F(t) &\equiv \sum_{h=0}^{l} L_h t^h \exp \lambda^{(h)} t + \sum_{i=0}^{m} M_i \exp \mu^{(i)} t, \\
G(t) &\equiv \sum_{j=0}^{n} N_j t^j \exp \nu^{(j)} t + \sum_{k=0}^{r} R_k \exp \rho^{(k)} t.
\end{aligned} \tag{16}
$$

Writing $Q(t) \equiv \exp \theta t$, thus defining θ, we have

$$t^h = h! \exp \delta^{(h)} t, \qquad t^j = j! \exp \delta^{(j)} t;$$

and equating coefficients of θ^u in the identity in t, $Q(t)G(t) \equiv F(t)$, we get (12). Again,

$$e^{\xi t} Q(t) = e^{(\xi+\theta)t} \equiv [e^{\xi t} P(t)]/G(t);$$

hence

$$G(t) \exp A(\xi, \theta) t = F(t) \exp \xi t;$$

whence (15) follows. The duals are obtained by the substitution $t \to \log (1+t)$, obviously permissible under the convergence assumptions, in the foregoing identities, and (3), (5). By the inverse substitution $t \to e^t - 1$, the equivalence of the duals follows. To remove the convergence restriction we refer to a former paper* where it was proved that in such formal uses of power series as are sufficient here convergence is irrelevant.

To save space, all applications, of which several have been made, will be omitted here, but will appear elsewhere.

CALIFORNIA INSTITUTE OF TECHNOLOGY

---

* Transactions of this Society, vol. 25 (1923), pp. 135–154, §2.

DECOMPOSITION OF ELEMENTS IN ABELIAN GROUPS*

RUFUS OLDENBURGER

1. **Introduction.** Let G be an abelian group of elements g with operation $+$ (sum), and unit element 0 (zero). We shall be concerned with the following property:

PROPERTY P_n. *There exist n distinct elements in G such that their sum vanishes.*

In the present paper we determine necessary and sufficient conditions for an abelian group to have the property P_n. The author first proved the validity of these conditions for fields, but, as noted by T. Nakayama, the operation $\times$ does not occur, so that the theory may be stated for a system of elements with only one operation $+$ defined for these elements. Groups with the property P_n have useful algebraic applications.†

One is naturally led to consider the decomposability of any given element of a group G into a sum of distinct elements of G. This problem is treated in §3.

It is necessary in the treatment of decompositions of elements to distinguish only between nonzero elements of order 2, elements of order different from 2, and the zero element.

2. **Groups with property P_n.** A pair (h, k) of elements h, k in a group G satisfying

$$h + k = g$$

is called a *g-pair* in G. If one or both of the elements h, k is zero, the pair (h, k) is called a *null g-pair*.

An element q in G of order 2 $(q+q=0)$ will be said to be *singular*. The remaining elements of G are said to be *nonsingular*.

---

* Presented to the Society, September 6, 1938.

† Let the minimal number $m(F)$ of a form F of degree r, with respect to a field K^*, designate the smallest value of σ for which F can be written as a sum $\lambda_1 L_1^r + \cdots + \lambda_\sigma L_\sigma^r$, where the λ's are in K^*, and the L's are linear forms with coefficients in K^*. Let K denote a field whose characteristic and order are such that the symmetric q-way and $(q+1)$-way matrices of the forms Q and QL of degree q and $(q+1)$, respectively, are unique, the coefficients being in K. We can prove that if K has the property P_{q+1}, the following inequality is true:

$$m(QL) \leqq (q+1)m(Q),$$

where the m's denote minimal numbers with respect to K.

We shall use the following obvious lemmas in the development of this paper:

LEMMA 1. *The nonsingular elements of an abelian group may be grouped into distinct* 0*-pairs.*

LEMMA 2. *The singular elements of an abelian group G may be grouped into distinct q-pairs for each nonvanishing singular q in G.*

LEMMA 3. *The four elements of two distinct* 0*-pairs of nonsingular elements in an abelian group are mutually distinct.*

LEMMA 4. *The four elements in two distinct q-pairs, $(q \neq 0)$, of singular elements in an abelian group are mutually distinct.*

It is to be observed that the null q-pair is $(0, q)$.

If there are two or more singular elements in an abelian group G, these elements form a subgroup of G, which we shall call the *singular subgroup* G' of G. If $G = G'$, the group G is said to be *singular*.

If $g = g_1 + \cdots + g_n$, where $g_1, \cdots, g_n$ are in G, we shall say that $(g_1, \cdots, g_n)$ is an *n-representation* of g. If $g_1, \cdots, g_n$ are distinct we shall call the representation *proper*.

We shall prove the following theorem where it is naturally understood that the order of the group is not less than n:

THEOREM 1. *An abelian group G possesses Property P_n except in the following cases:*

(a) *$n = 2$, and G is singular;*

(b) *G is singular, of finite order m, and $n = m - 2$;*

(c) *G possesses a singular subgroup of order 2, G is of finite order, and n equals the order of G.*

We let G be singular. If for a pair of elements a, b in G we have $a + b = 0$, then $a = b$; whence it follows that G does not possess Property P_2. In case (b), $m \geqq 4$. The elements of G may be paired into an even number of distinct q-pairs for any given nonzero element q in G, whence the sum of all of the elements in G is zero. If the sum of $m - 2$ distinct elements of G vanishes, G possesses Property P_2, which has been proved to be impossible.

If part (c) of Theorem 1 is satisfied, G possesses a singular subgroup G' of elements 0, q. By Lemma 1 the sum of elements of G not contained in G' is zero; whence the sum of all elements in G is q. Thus Property P_n does not hold in case (c).

It remains to show that if (a), (b), and (c) are not satisfied by G and n, the group G possesses Property P_n.

It is to be remarked that if G is infinite and contains at least one nonsingular element, it contains an infinite number of such elements.

Let d denote the number of nonsingular elements in G, where $\infty \geqq d \geqq 0$. Assume $d>0$, and $n \leqq d$. If $n=2k$, we obtain a proper n-representation of zero by taking k 0-pairs of nonsingular elements; if $n=2k+1$, we adjoin 0 to the k 0-pairs just mentioned.

We assume now that d is finite and that $n=d+s$, where in turn $s=4k+i$, $(i=1, 2, 3, 4, k \geqq 0)$. The group G now has a singular subgroup G' whose order we shall denote by m', where m' is finite. If $i=1$, we obtain a proper n-representation of zero by adjoining zero to the sum of the nonsingular elements of G, and the elements in $2k$ distinct non-null q-pairs, $(q \neq 0)$, of G'. If $i=3$, we adjoin q to the elements in $(2k+1)$ non-null q-pairs of singular elements, $(q \neq 0)$, and the nonsingular elements of G. If $i=4$, we adjoin $(2k+2)$ q-pairs, $(q \neq 0)$, of singular elements to the nonsingular elements of G.

Let $i=2$, and $k=0$. Since it is assumed that (a) is not satisfied, we have $d>0$. Since (c) is not satisfied, $m' \geqq 4$. We adjoin two distinct q-pairs, $(q \neq 0)$, from G' to $(d-2)/2$ distinct 0-pairs of nonsingular elements. It remains to take $i=2$, $k \geqq 1$. We can now write $s=4k'+6$, where $k' \geqq 0$. Since the order m' of G' is finite, it is even. We assume that $s \neq m'-2$. Since s is even, $s \leqq m'-4$; whence

$$m' \geqq 4k' + 10. \tag{1}$$

Actually $m' \geqq 4k'+12$, but this fact is not needed. It follows from (1) that the order m' of G' is at least 2^4. Select from G' three distinct non-null q-pairs, $(q \neq 0)$,

$$(q_1, q_2), (q_3, q_4), (q_5, q_6) \tag{2}$$

such that $q_1+q_3=q_5$, whence

$$q_1 + q_3 + q_5 = 0. \tag{3}$$

It is readily seen from $-q_i=q_i$ that q_1+q_3 is different from the elements in the pairs $(0, q)$, (q_1, q_2), (q_3, q_4); so q_5 actually occurs in a q-pair distinct from these. Let (q_7, q_8) be a q-pair in G' which is non-null and distinct from (2). Then

$$q + q_7 + q_8 = 0. \tag{4}$$

By (3) and (4) we have

$$q + q_1 + q_3 + q_5 + q_7 + q_8 = 0. \tag{5}$$

We obtain a proper $(d+4k'+6)$-representation of zero by adjoining to the sum exhibited in (5) the nonsingular elements of G and the

elements of $2k'$ distinct q-pairs in G' which are non-null and different from the pairs (2) and (q_7, q_8). That this is possible follows from (1) and the fact that the pairs (2), (q_7, q_8), and $(0, q)$ contain exactly 10 elements.

If, finally, $i=2$ and $s=m'-2$, since the conditions of (b) are assumed to be not satisfied, there exist nonsingular elements in G, whence $d>0$. We have $m=4l$. We obtain a proper n-representation of 0 by taking all elements of G except one 0-pair of nonsingular elements.

COROLLARY 1. *An abelian group G of infinite order possesses the Property P_n, except when G is singular and $n=2$.*

COROLLARY 2. *An abelian group G with no singular subgroup possesses Property P_n for each n.*

COROLLARY 3. *A field of characteristic p and order m (finite or infinite) possesses the Property P_n if and only if $p\neq 2$, or if $p=2$, $n\neq 2$, $m-2$.*

COROLLARY 4. *The sum of the elements of a finite abelian group G is zero except when G contains a singular subgroup G' of order 2; in the latter case it is equal to the nonzero element in G'.*

COROLLARY 5. *The sum of elements of a finite field K of characteristic p and order m is zero except when $p=m=2$.*

One can prove that the sum of the elements of a field K with characteristic different from 2 is zero rather simply and directly in another manner. Let $q_1, \cdots, q_m$ denote the distinct elements of K. Let $q_1+\cdots+q_m=q$, q being in K. The set $-q_1, \cdots, -q_m$ is obviously the same as the set $q_1, \cdots, q_m$. Hence $q=0$.

3. **Decomposability of nonzero elements of abelian groups.** It will appear from the development of this section that the *existence of a proper n-representation of a nonsingular element of an abelian group G implies the existence of a proper n-representation of each nonsingular element* of G. Similarly, the existence of a proper n-representation of a nonzero singular element of G implies the existence of a proper n-representation for each such element of G. We therefore make the following definitions:

PROPERTY P_n'. *There exists a proper n-representation of each nonsingular element in G.*

PROPERTY P_n''. *There exists a proper n-representation of each nonzero singular element in G.*

THEOREM 2. *An abelian group G containing nonsingular elements possesses the property P_n' unless n equals the order of the group.*

By Corollary 4 of §2, if n is the order of the group, there is no proper n-representation of a nonsingular element in G. In what follows we therefore assume that n is less than the order of the group.

As above, we let d denote the number of nonsingular elements of G, where now $\infty \geqq d \geqq 0$. Assume $n \leqq d$. We obtain a proper n-representation of a nonsingular element g by taking a sufficient number of 0-pairs of nonsingular elements different from $(g, -g)$ and adjoining g or 0, g according as n is odd or even.

We assume now that d is finite, and $n = d+s$, $(s>0)$. Let m denote the order of the singular subgroup G'. Since d is finite, m is finite. Let the nonsingular elements of G be denoted by $g_1, \cdots, g_d$. Since $s<m$, there exist at least a set of distinct elements $q_1, \cdots, q_{s+1}$ in G'. For some element q_α in G' we have the equality

$$g_1 + \cdots + g_d + q_1 + \cdots + q_{s+1} = q_\alpha. \tag{6}$$

The equality

$$q_\alpha + x = g,$$

where g is nonsingular, is satisfied for a nonsingular element g_i in G. There is a value of j such that $g_j = -g_i$. Adding g_i to both sides of (6) we obtain the proper $(d+s)$-representation of g:

$$(g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_d, q_1, \cdots, q_{s+1}).$$

THEOREM 3. *An abelian group G containing non-null singular elements possesses the property P_n'' unless n is the order of G and the order of the singular subgroup G' of G is greater than 2.*

By Corollary 4, if G is finite of order n, there is a proper n-representation of a non-null singular element of G if and only if the singular subgroup G' of G is of order 2.

In what follows we assume that n is less than the order of G. The group G contains a singular subgroup G' of order m, where $m \geqq 2$. Let the order of G be $d+m$, where $\infty \geqq d \geqq 0$. If $n \leqq d$, we obtain a proper n-representation of a nonzero singular element q by choosing elements from the pair $(0, q)$, and 0-pairs of nonsingular elements. If $n = d+4k+i$, where $k \geqq 0$ and $i = 1, 2$, or 3, we obtain a proper n-representation of q by taking all nonsingular elements of G, a sufficient number of non-null q-pairs from G', and elements from the null pair $(0, q)$. It remains to consider the case where $n = d+s$, $s = 4k+4$. Since $m = 4l$, and $s \leqq m-4$, we have

$$4k \leqq m - 8. \tag{7}$$

We choose distinct non-null q-pairs of elements of G' given by

$$(q_1, q_2), (q_3, q_4), (q_5, q_6) \tag{8}$$

so that $q_1+q_3=q_5$. By (7) there are at least $2k$ distinct non-null q-pairs $P_1, \cdots, P_{2k}$ in G' different from (8). We obtain a proper n-representation of q by adding q, q_1, q_3, q_5 to the elements in $P_1, \cdots, P_{2k}$ and the nonsingular elements of G.

4. **Construction of representations.** We used very special proper representations in §§2 and 3 to prove the existence of proper representations. In the present section we show how, under very general circumstances, proper representations of a given element g may be obtained from proper representations of other elements in the group.

THEOREM 4. *Let S be a set of n distinct elements in an abelian group G, and let h denote the sum of the elements in S. Let g be an element of G distinct from h, and let p be the order of k, where $k=g-h$. The element g has a proper n-representation P containing at least $n-1$ elements of S unless $p\neq 0$ and the elements of S can be grouped into cosets of the cyclic subgroup $K=[0, k, 2k, \cdots, (p-1)k]$.*

Let $g_1, \cdots, g_n$ be n distinct elements of G whose sum is h. If for some i we have

$$g_i + k \neq g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n,$$

the representation $P_i=(g_1, \cdots, g_{i-1}, g_i+k, g_{i+1}, \cdots, g_n)$ is a proper n-representation of g. Assume therefore that for each i we have g_i+k equal to one of the quantities $g_1, \cdots, g_{i-1}, g_{i+1}, \cdots, g_n$. Since $g_i\neq g_j$ for $i\neq j$, the set $(g_1+k, \cdots, g_n+k)$ is equal to the set $(g_1, \cdots, g_n)$. We can evidently order the subscripts on the g's and choose r, s, so that

$$g_r + k = g_1, \quad g_i + k = g_{i+1}, \qquad i = 1, \cdots, r-1, \tag{9}$$

and

$$g_{r+s} + k = g_{r+1}, \quad g_j + k = g_{j+1}, j = r+1, \cdots, r+s-1,$$

and so on. Adding the equations (9) we obtain

$$\sum_{i=1}^{r} g_i + rk = \sum_{i=1}^{r} g_i. \tag{10}$$

If p does not divide r, the formula (10) implies that $k=0$, contradicting $k\neq 0$. Hence r, s, and so on, are each divisible by p, so

that p is not zero. Let $r=qp$ where $q>1$. By (9) $g_{i+1}=g_1+ik$ for $i=1, 2, \cdots, qp-1$. Hence $g_{p+1}=g_1+pk=g_1$, contradicting $g_1 \neq g_{p+1}$. It follows that the g's may be grouped into cosets of K, whence the theorem is proved.

COROLLARY 6. *Let S be a set of n distinct elements $g_1, \cdots, g_n$ of an abelian group G, where the elements of S are not the elements comprising cosets of a cyclic subgroup of G. Let the sum of the elements in S be denoted by h. For each element g in G there is a proper representation $(g_1, \cdots, g_{i-1}, g_i+k, g_{i+1}, \cdots, g_n)$ of g where $k=g-h$.*

THE INSTITUTE FOR ADVANCED STUDY

THE MAXIMUM NUMBER OF DISTINCT CONTACTS OF TWO ALGEBRAIC SURFACES*

T. R. HOLLCROFT

1. **Introduction.** For more than half a century, it has been known that the maximum number of distinct contacts of two algebraic surfaces whose intersection curve is irreducible is the genus of that curve.†

The number of contacts of two surfaces whose entire intersection curve consists of straight lines has been found.‡ This is the maximum number of contacts of two surfaces of given orders.

The purpose of this paper is to obtain the maximum number of distinct contacts of two algebraic surfaces when the intersection curve consists of any given number of components of given orders and genera. Cases are treated in which the two surfaces have singular points or singular curves in common.

2. **Method.** From any point of S_3 not on the developable of C, a space curve C with h apparent double points projects into a plane curve C' with h nodes. Since C' may have p additional nodes, where p is the genus of both C and C', the space curve C may have p nodes. The necessary and sufficient condition for the two surfaces to have a contact is for C to have a node.§

If C is irreducible and is the complete intersection of two surfaces M and N, the maximum number of contacts of M and N is the genus

* Presented to the Society, December 28, 1937.

† E. Pascal, Repertorium der höheren Mathematik, vol. 2 (1902), p. 225. In a later supplementary volume of the Repertorium (vol. 2_2 (1922), pp. 653–656), more than two pages are devoted to the topic, "Contacts of two surfaces," but the problem of this paper is not treated.

‡ Encyklopädie der mathematischen Wissenschaften, vol. 3, C, 9, (1926), p. 1277.

§ E. Pascal, Repertorium der höheren Mathematik, vol. 2_2 (1922), p. 654.

of C. This is the case referred to in the first paragraph of the preceding section.

If C is reducible, M and N have a contact at each simple intersection of the components of C (which count as nodes of C) and also at each node of any component. The maximum number of possible contacts is, therefore, the total sum of the genera of all the components and the number of intersections of the components, taken two at a time. This sum is to be found when C consists of any given number of components of given orders and genera, and when the two surfaces have singular points or curves in common.

3. **Surfaces with no common singularities.** Let M and N be two algebraic surfaces of orders μ and ν with no singularities in common. These surfaces intersect in a curve C of order $n=\mu\nu$ which consists of α components C_j, of orders n_j, genera p_j, and with h_j apparent double points, respectively.

Any two components C_j and C_k intersect in i_{jk} points and have I_{jk} apparent intersections with $i_{jk}+I_{jk}=n_jn_k$. The total number of intersections of the α components of C is $\frac{1}{2}\sum i_{jk}$, $(j\neq k;\ j,\ k=1,\cdots,\alpha)$. A similar expression gives the total number of apparent intersections of all the components, and there results the relation

$$\sum i_{jk}+\sum I_{jk}=\sum n_jn_k, \qquad j\neq k;\ j,\ k=1,\cdots,\alpha. \tag{1}$$

The apparent intersections of the components count as apparent double points of the total intersection C. This is expressed by the relation

$$\sum h_j+\tfrac{1}{2}\sum I_{jk}=\tfrac{1}{2}\mu\nu(\mu-1)(\nu-1). \tag{2}$$

The sum of the genera of the components is

$$\sum p_j=\tfrac{1}{2}\sum(n_j-1)(n_j-2)-\sum h_j. \tag{3}$$

The maximum number T of distinct contacts of M and N is, therefor, $T=\frac{1}{2}\sum i_{jk}+\sum p_j$. Substituting in this expression the value of $\frac{1}{2}\sum i_{jk}$ obtained from (1) and (2) by eliminating $\frac{1}{2}\sum I_{jk}$ and the value of $\sum p_j$ given in (3), we obtain

$$T=\tfrac{1}{2}\mu\nu(\mu+\nu-4)+\alpha. \tag{4}$$

This remarkably simple result establishes the following theorem:

The maximum number T of distinct contacts of two algebraic surfaces is independent of the orders and genera of the components of the intersection curve, and, moreover, T is increased by one and only one for each additional component.

Of the total number T of possible contacts, the surfaces M and N *have* $\frac{1}{2}\sum i_{jk}$ contacts and *may have* $\sum p_j$ additional contacts. Each of these two numbers is a function of both the orders and genera of the components of C, but their sum T is not.

From the expression (4), it appears that the greatest possible number of distinct contacts of two surfaces of orders μ and ν occurs when the number of components α is a maximum. The maximum value of α is $\mu\nu$; that is, the intersection consists entirely of lines. The number of contacts in this case is

$$\tfrac{1}{2}\mu\nu(\mu + \nu - 4) + \mu\nu = \tfrac{1}{2}\mu\nu(\mu + \nu - 2).^*$$

This maximum number of contacts for two surfaces of given order not only can be, but must be attained when C consists of $\mu\nu$ lines.

The developable surface of C is of order $\mu\nu(\mu+\nu-2)$. When C consists of $\mu\nu$ lines, this surface degenerates into $\frac{1}{2}\mu\nu(\mu+\nu-2)$ distinct planes, each counted twice. Each of these planes is determined by a pair of intersecting lines of the $\mu\nu$ lines common to M and N. Each of the $\frac{1}{2}\mu\nu(\mu+\nu-2)$ planes is, therefore, tangent to both M and N at the intersection of the line pair determining that plane.

4. **Application to quadrics.** If M and N are quadrics, $\mu=\nu=2$ and $T=\alpha$; that is, the number of possible contacts is always equal to the number of components of the intersection curve.

The only pairs of proper quadrics in S_3 that do not have contact with each other are those whose intersection is an elliptic quartic. Also, all pairs of proper quadrics, whose intersection consists of $1 \leqq \alpha \leqq 4$ distinct *rational* components, *have* α contacts. If $\alpha \geqq 2$, all components must be rational.

The maximum number of distinct contacts of two quadrics is four, which occurs when the quadrics intersect in four lines. In this case, the equations of M and N may be written†

$$M\colon\ \lambda_1(x_1^2 + x_2^2) + \lambda_2(x_3^2 + x_4^2) = 0,$$
$$N\colon\ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

The equation of the developable surface of the complete intersection of these two quadrics is found to be

$$(x_1^2 + x_2^2)(x_3^2 + x_4^2) = 0.$$

---

* Encyklopädie, loc. cit. This expression may be derived directly. A set of $\mu\nu$ coplanar lines has $\frac{1}{2}\mu\nu(\mu\nu-1)$ intersections. When the $\mu\nu$ lines constitute the total intersection of M and N, $h=\frac{1}{2}\mu\nu(\mu-1)(\nu-1)$ intersections are apparent, and the remaining $\frac{1}{2}\mu\nu(\mu\nu-1)-h=\frac{1}{2}\mu\nu(\mu+\nu-2)$ are actual intersections of the $\mu\nu$ lines.

† V. Snyder and C. H. Sisam, *Analytical Geometry of Space*, 1917, p. 163.

This surface consists of four planes, each of which passes through a pair of lines common to M and N.

5. **Surfaces with singular points in common.** Let M and N have in common β points P_i which are r_i-fold on M and s_i-fold on N. Three cases will be considered.

CASE 1. None of the β points P_i lie at an intersection of the α components of C.

The complete intersection curve C has

$$h = \tfrac{1}{2}\mu\nu(\mu - 1)(\nu - 1) - \tfrac{1}{2}\sum r_i s_i(r_i - 1)(s_i - 1), \qquad i = 1, \cdots, \beta.^*$$

This reduction in h applies only to bisecants of the individual components and not to those of pairs of components.

The surface C has β multiple points of orders $r_i s_i$ which occur at distinct points on components of C. Hence the sum $\sum^{\alpha} p_j$ of the genera of the components C_j is reduced by

$$\tfrac{1}{2}\sum r_i s_i(r_i s_i - 1) - \tfrac{1}{2}\sum r_i s_i(r_i - 1)(s_i - 1) = \tfrac{1}{2}\sum r_i s_i(r_i + s_i - 2), \qquad i = 1, \cdots, \beta.$$

Substituting this reduction in (4), we obtain

$$T = \tfrac{1}{2}\mu\nu(\mu + \nu - 4) + \alpha - \tfrac{1}{2}\sum^{\beta} r_i s_i(r_i + s_i - 2). \tag{5}$$

The number of intersections $\tfrac{1}{2}\sum i_{jk}$, $(j \neq k;\ j,\ k = 1, \cdots, \alpha)$, is not affected by the β multiple points in this case.

There is no relation between α and β. Two or more multiple points may be on one component, so that β may be larger than α. The only restriction on the multiple points is that their size and number must not be so great as to cause any component to break up into curves of lower order intersecting at certain of the points P_i.

CASE 2. The β multiple points P_i which are r_i-fold on M and s_i-fold on N are so situated that two and only two component curves C_j and C_k of C pass through each P_i. The composite multiplicity of P_i on C_j and C_k is $r_i s_i$. Let the multiplicity of P_i on C_j be ρ_j and on C_k, ρ_k. Then

$$\rho_j + \rho_k = r_i s_i.$$

The components C_j and C_k have $\rho_j \rho_k$ intersections with each other at P_i in addition to $\tfrac{1}{2}\rho_j(\rho_j - 1)$ intersections of branches of C_j and $\tfrac{1}{2}\rho_k(\rho_k - 1)$ intersections of branches of C_k, that is,

* N. Salvatore-Dino, *Sul genere delle curve gobbe*, Rendiconti di Napoli, vol. 18 (1879), pp. 132–136.

$$\rho_j\rho_k + \tfrac{1}{2}\rho_j(\rho_j - 1) + \tfrac{1}{2}\rho_k(\rho_k - 1) = \tfrac{1}{2}r_is_i(r_is_i - 1).$$

The $\rho_j\rho_k$ intersections of C_j and C_k at P_i count among the i_{jk} intersections of C_j and C_k, but since these $\rho_j\rho_k$ intersections lie at P_i, they are associated with no contacts of M and N. The number of contacts of M and N that would occur at intersections of C_j and C_k is, therefore, reduced by $\rho_j\rho_k$.

The genus p_j of C_j is

$$p_j = \tfrac{1}{2}(n_j - 1)(n_j - 2) - h_j - \tfrac{1}{2}\rho_j(\rho_j - 1),$$

and similarly for the genus p_k of C_k. The genera of component curves that do not pass through points P_i are not affected. Since there are β points P_i, it follows that β curves C_j and β curves C_k are affected. Hence,

$$T = \tfrac{1}{2}\mu\nu(\mu + \nu - 4) + \alpha - \tfrac{1}{2}\sum^{\beta}\rho_j(\rho_j - 1) - \tfrac{1}{2}\sum^{\beta}\rho_k(\rho_k - 1) - \tfrac{1}{2}\sum^{\beta}\rho_j\rho_k, \qquad j \neq k,$$

which reduces to

$$T = \tfrac{1}{2}\mu\nu(\mu + \nu - 4) + \alpha - \tfrac{1}{2}\sum^{\beta}r_is_i(r_is_i - 1). \tag{6}$$

It has been assumed that a given component C_j may have a point P_i at its intersection with one component C_k, but not another multiple point at its intersection with a third component. This restriction is not necessary, however, since the reduction in the sum $\sum p_j + \frac{1}{2}\sum i_{jk}$ is the same whether two points P_i lie on two distinct pairs of components, or at two of the intersections of one component with two others, or at two intersections of the same two components. The same argument holds for three or more points P_i.

CASE 3. In this case, β_1 points P_i satisfy the conditions of Case 1 and β_2 of Case 2. The resulting value of T is a combination of formulas (5) and (6):

$$T = \tfrac{1}{2}\mu\nu(\mu + \nu - 4) + \alpha - \tfrac{1}{2}\sum^{\beta_1}r_is_i(r_i + s_i - 2) - \tfrac{1}{2}\sum^{\beta_2}r_is_i(r_is_i - 1).$$

6. **Surfaces with a common contact curve.** If M and N have simple contact along a curve C_1, then at nodes of C_1 or at intersections of the other components of f with C_1, M and N have contact of higher order than at ordinary points of C_1. Since all such contacts occur on C_1, they are not distinct contacts, and will not be counted in the maximum number of distinct contacts.

If C consists of α components C_j, $(j = 1, \cdots, \alpha)$, of which α_1 com-

ponents $C_1, \cdots, C_{\alpha_1}$ are contact curves, the maximum number of distinct contacts of M and N, that is, the maximum number T of contacts at points not on any of the contact components, is $T=\sum p_j+\frac{1}{2}\sum i_{jk}$, $(j\neq k;\ j,\ k=\alpha_1+1,\ \alpha_1+2, \cdots, \alpha)$. This does not reduce to a function of μ, ν, and α as is the case when all of the components are simple.

7. **Surfaces with a common multiple curve.** Let the total intersection of M and N consist of two components C_1 and C_2 of orders n_1, n_2 and genera p_1, p_2, such that C_1 is i_1-fold on M and i_2-fold on N. The contacts of M and N at nodes of C_1 and C_2 and at intersections of C_1 and C_2 will be distinct contacts as before. If t is the number of intersections of C_1 and C_2, the maximum number of distinct contacts of M and N is p_1+p_2+t.

The value of this expression in terms of μ, ν, i_1, i_2, n_1, p_1 is obtained by using formulas (A) and (3) of a former paper.*

More generally, if M and N intersect in α components, C_j of orders n_j and genera p_j, and if one component C_1 is i_1-fold on M and i_2-fold on N and the remaining components are simple on both surfaces, the maximum number of contacts of M and N is

$$\tfrac{1}{2}(i_1+i_2-3)\left[2i_1i_2(2n_1+p_1-1)-n_1(i_2\mu+i_1\nu)\right] + (\mu\nu-i_1i_2n_1)(\mu+\nu-4)+p_1+\alpha-1.$$

Thus, in this case, the maximum number of contacts of M and N can be obtained when the orders of the two surfaces, the number of components, and the order, genus, and multiplicities of the multiple curve are known.

8. **Stationary contacts of two surfaces.** The maximum number of distinct stationary contacts of two surfaces can be obtained by similar methods when the problem of finding the maximum number of cusps of a plane curve of given order and genus has been solved.

WELLS COLLEGE

---

* T. R. Hollcroft, *Characteristics of multiple curves and their residuals*, this Bulletin, vol. 39 (1933), pp. 960–961.

ON BOUNDS FOR PARAMETERS IN n-NODED SOLUTIONS OF STURM-LIOUVILLE EQUATIONS*

J. K. L. MacDONALD

Generalized Sturm-Liouville differential equations† are of considerable importance, particularly in applied mathematics. Such equations have the form

$$\left(\frac{d}{dx}\,p(x,\lambda)\,\frac{d}{dx}+q(x,\lambda)\right)y(x,\lambda)=0 \tag{1}$$

in which (in probably the most important cases) the independent variable x ranges over a continuous finite or infinite real interval (a, b) and in which values for the parameter λ are implicitly determined by continuity and end conditions for y and (dy/dx). Fairly relaxed conditions are known‡ to be sufficient to ensure the existence of a discrete set of λ values λ_n, $(n=n_0, n_0+1, \cdots)$, for which λ_n corresponds to a unique solution $y=y_n$ with exactly n nodes in $a<x<b$, a and b possibly being real singularities for the equation.

In this paper we shall assume that such conditions are satisfied and, in particular, that at least in the open interval (a, b) the functions $p>0$, y, and (dy/dx) are continuous, and that (dp/dx) and q are piecewise continuous functions of x. (We shall use the term *piecewise continuity* of $f(x)$ to imply continuity at least in an open interval concerned, except possibly for a finite number of points at which $f(x)$ is equal to one of the bounded unequal limits $f(x-0)$ and $f(x+0)$. Correspondingly we shall use (d/dx) to denote left- or right-hand differentiation whichever is called for in a given expression.) We shall discuss methods for bounding λ_n and

$$\cot^{-1}\left((p/q)^{1/2}(d/dx)\log(pq)^{1/4}y_n\right)$$

by use of a phase angle ϕ, and shall stress the accuracy of asymptotic forms§ (as $n\to\infty$), a feature which has been little investigated in the past.

Let us consider an open interval (α, β) contained by (a, b), and let

---

* Presented to the Society, September 7, 1937.

† See McCrea and Newing, Proceedings of the London Mathematical Society, (2), vol. 37 (1933), p. 520, for general discussion and references.

‡ McCrea and Newing, loc. cit., p. 520.

§ See Langer, this Bulletin, vol. 40 (1934), p. 545, and Dunham, Physical Review, vol. 41 (1932).

us introduce disposable continuous functions $r>0$* and s with piecewise continuous x-derivatives r' and s'. We define a dependent continuous function (phase angle) ϕ by the relation†

$$\cot \phi = s + (rpy'/y). \tag{2}$$

Then we can eliminate y'/y between equations (1) and (2) to get the differential equation satisfied by ϕ:

$$\begin{aligned} \frac{d\phi}{dx} = \frac{1}{rp} + \left(rq - \frac{1}{rp} + \frac{s^2}{rp} + \frac{r's}{r} - s'\right) \sin^2 \phi \\ - \left(\frac{r'}{2r} + \frac{s}{rp}\right) \sin 2\phi. \end{aligned} \tag{3}$$

Our main concern is to choose the functions r and s so that equation (3) can be readily solved analytically or so that a comparison theorem can be used to bound ϕ closely between known forms. The field of choice is very wide and encourages individual ingenuity in special cases; so we shall concentrate on what appear to be the most useful general forms. Note that we can determine boundary values for ϕ by use of (2) and the asymptotic forms of y'/y near the boundaries, together with the property that ϕ passes through a multiple of π only as x passes through a nodal value for y, ϕ being an increasing function of x near each node. Further we may note the useful suggestion that in general‡

$$\int_\alpha^\beta (q/p)^{1/2} dx \sim n\pi \quad \text{as} \quad n \to \infty, \tag{4}$$

where n is the number of nodes in the interval (α, β) over which $q \geqq 0$ and over which the integral exists. In fact one of our main objects will be to determine the degree of accuracy of asymptotic relations like (4).

If we set $s = -pr'/2$, we rid (3) of the indefinite signed term $\sin 2\phi$:

$$\frac{d\phi}{dx} = \frac{1}{pr} + \left(rq - \frac{1}{pr} - \frac{p(r')^2}{4r} + \frac{(pr')'}{2}\right) \sin^2 \phi. \tag{5}$$

---

* This condition may be relaxed to allow $r \to 0$, $r \to \infty$, or r to be only piecewise continuous within (α, β), but then it may be difficult to determine the possible discontinuities in ϕ.

† Prüfer, Mathematische Annalen, vol. 95 (1926), pp. 499–518, was the first to make general use of a polar angle. His form corresponds to setting $s=0$ and $r=1$ in our general expression.

‡ Langer, loc. cit., Dunham, loc. cit.

In particular, we can take (α, β) as an open interval over which $pq>0$* and over which (pq) and $(pq)'$ are continuous and $(pq)''$ piecewise continuous. Then we can set $r=(pq)^{-1/2}$ and get, as a result, the form

$$\frac{d\phi}{dx}=\left(\frac{q}{p}\right)^{1/2}(1+F\sin^2\phi), \tag{6}$$

with

$$\cot\phi=\left(\frac{(pq)'}{4pq}+\frac{y'}{y}\right)\left(\frac{p}{q}\right)^{1/2},$$

$$F=\frac{5}{16pq^3}\left(\left(\frac{d}{dx}pq\right)^2-\frac{4}{5}\left(q\frac{d}{dx}p\frac{d}{dx}pq\right)\right).$$

We have much reason to regard equation (6) as our most simple and powerful form for discussing many types of examples, and in particular for determining the accuracy of the relation (4). For such purposes we can use the following among the many inequalities derivable directly from (6) or through a comparison theorem.† We take (c, d) to be any appropriate interval contained by (α, β), with ϕ_c and ϕ_d the values for ϕ at $x=c$ and $x=d$, respectively. Then we obtain the relations

$$\int_c^d\left(\frac{q}{p}\right)^{1/2}dx\leqq\phi_d-\phi_c\quad\left(\int_c^d\left(\frac{q}{p}\right)^{1/2}dx\geqq\phi_d-\phi_c\right) \tag{7a}$$

if $0\leqq F$ $(0\geqq F)$ over the interval (c, d), which is a special case of the stronger form

$$I\leqq\int_c^d\left(\frac{q}{p}\right)^{1/2}dx\leqq I-\int_c^d\left(\frac{q}{p}\right)^{1/2}\left(\frac{F-k}{1+k}\right)dx$$

$$\left(I\geqq\int_c^d\left(\frac{q}{p}\right)^{1/2}dx\geqq I-\int_c^d\left(\frac{q}{p}\right)^{1/2}\left(\frac{F-k}{1+k}\right)dx\right) \tag{7b}$$

if $k>-1$ is an upper (a lower) bound for F over (c, d), where

$$I=\int_{\phi_c}^{\phi_d}\frac{d\phi}{1+k\sin^2\phi}=\frac{1}{2(1+k)^{1/2}}\sin^{-1}\frac{1-(2+k)\sin^2\phi}{1+k\sin^2\phi}\Bigg|_{\phi_c}^{\phi_d}$$

(evaluated as an increasing function of ϕ_d). Furthermore

* See the first footnote on p. 165.

† See, for example, Kamke's *Differentialgleichungen*, pp. 82–86. We use the generalized form with $\leqq$ replacing $<$, and with $\phi(\xi)\leqq\psi(\xi)$ in Kamke's theorem on p. 82.

$$\int_c^d \left(\frac{q}{p}\right)^{1/2}(1+F)dx \leqq \phi_d - \phi_c$$

(8a)

$$\left(\int_c^d \left(\frac{q}{p}\right)^{1/2}(1+F)dx \geqq \phi_d - \phi_c\right)$$

if $F \leqq 0$ ($F \geqq 0$) over (c, d), which is a special case of the general form

$$\int_c^d \left(\frac{q}{p}\right)^{1/2}(1+\gamma^2 F)dx \leqq \phi_d - \phi_c$$

(8b)

$$\left(\int_c^d \left(\frac{q}{p}\right)^{1/2}(1+\gamma^2 F)dx \geqq \phi_d - \phi_c\right)$$

if $\gamma^2 F \leqq F \sin^2\phi$ ($\gamma^2 F \geqq F \sin^2\phi$) over (c, d), where γ is some prescribed function usually suggested by the asymptotic form of $\sin^2\phi$ near the end points c and d. We can usually find suitable asymptotic forms by use of $\cot\phi$ in (6) and the Riccati form for (1), namely,

$$\frac{d}{dx}p\left(\frac{y'}{y}\right) + p\left(\frac{y'}{y}\right)^2 + q = 0. \tag{9}$$

Let us illustrate the use of the above inequalities. For examples in which $(q/p)^{1/2}$ and $(q/p)^{1/2}F$ are integrable over the interval between the end points at which the boundary conditions are given, we may readily bound λ_n, for instance, as in the following example. We consider the nonsingular equation $(d^2y/dx^2)+\lambda^2x^2y=0$ subject to $y=0$ at $x=1$ and at $x=2$. Here in (6) we have

$$F = \frac{3}{4\lambda^2x^4}, \qquad \frac{3}{4\lambda^2} \geqq F \geqq \frac{3}{64\lambda^2},$$

and $y'/y \sim 1/(x-1)+1/(x-2)$ near $x=1, 2$ (by (9) and the boundary conditions). Therefore, by $\cot\phi$ in (6) we have $\phi=0$ at $x=1$ and $\phi=(n+1)\pi$ at $x=2$, when n is the number of nodes *within* (1, 2). We therefore have the following bounding relations for $\lambda=\lambda_n$: From (7a),

$$\frac{3}{2}\lambda_n \leqq (n+1)\pi.$$

From the left inequality* in (7b),

$$(n+1)\pi\left(1+\frac{3}{4\lambda^2}\right)^{-1/2} \leqq \frac{3\lambda}{2} \leqq (n+1)\pi\left(1+\frac{3}{64\lambda^2}\right)^{-1/2};$$

* The right-hand side of (7b) yields very close but more complicated bounds.

whence, for example, in the case of $n-1$ nodes within (1, 2),

$$\frac{2}{3}\left(n^2\pi^2 - \frac{27}{16}\right)^{1/2} \leqq \lambda_{n-1} \leqq \frac{2}{3}\left(n^2\pi^2 - \frac{27}{256}\right)^{1/2}. \tag{10}$$

From (8a),

$$\frac{3\lambda_{n-1}}{2} + \frac{9}{32\lambda_{n-1}} \geqq n\pi,$$

which gives us a *larger* lower bound for λ_{n-1}. With little more trouble we could find even closer bounds for λ_{n-1}; for example, by dividing the interval (1, 2) into subintervals (c, d).

In examples for which there are points near which at least one of the expressions $(q/p)^{1/2}$ and $(q/p)^{1/2}F$ is not integrable we must, in general, use a comparison theorem to supplement the inequalities (7) and (8) in order to obtain *both* upper and lower bounds. Let us consider an example which involves singular end points and is not yet solved in closed form: $p=1-x^2$, $q=\lambda^2 x$, subject to the condition y bounded throughout the interval $-1<x<1$. In this case we have the intervals $(a, b)\equiv(-1, 1)$, $(\alpha, \beta)\equiv(0, 1)$, and, using (2) and the appropriate exponents at $x=\pm 1$, we obtain $\phi=0$ at $x=0$* and $\phi=(n+1)\pi$ at $x=1$. We introduce the constants

$$B = \int_0^1 \left(\frac{x}{1-x^2}\right)^{1/2} dx = \left(\frac{2}{\pi}\right)^{1/2} \Gamma^2\left(\frac{3}{4}\right) \doteq \frac{2}{3}\pi^{1/2}, \qquad C \doteq 1.71,$$

where C is the minimum value (in the interval $0<x<1$) of

$$\lambda^2 F = (3x^2-5)(x^2+1)/16x^3(x^2-1) \quad \text{at} \quad x^2 = (8/3)^{1/2} - 1.$$

Therefore from the left inequality in (7b) we get the bounding relation

$$\lambda_{n-1} \leqq ((n\pi/B)^2 - C)^{1/2} < n\pi/B.\dagger \tag{11}$$

Before considering such examples further let us note the following useful comparison‡ forms for use with (6):

$$(d\theta/dx) = (q/p)^{1/2}(1 + F\theta^2), \tag{12a}$$

---

* In this case py'/y has a finite nonzero value at $x=0$ if y is bounded at $x=-1$, because of the general identity for (1); namely $pyy'\big|_{x_1}^{x_2} = \int_{x_1}^{x_2}(p(y')^2 - qy^2)dx$ (whence $pyy'=0$ at most once in an interval where $q\leqq 0$ but $y'\not\equiv 0$ if $q\not\equiv 0$).

† In Legendre's equation $p=1-x^2$, $q=\lambda^2$, and $(a, b)\equiv(\alpha, \beta)\equiv(-1, 1)$. Here (7b) yields the inequality $\lambda_n^2<(n+1)^2-1/2$ for any solution with n nodes in $(-1, 1)$.

‡ See the second footnote on p. 166.

which is satisfied by $\theta = 4q(pq)^{1/2}/(pq)'$; and

$$(d\psi/dx) = (q/p)^{1/2}(1 + Ff(x)g(\psi - t)) \tag{12b}$$

which is satisfied by the form ψ given by the equation

$$\int \frac{d(\psi - t)}{g(\psi - t)} = \int \frac{f(x)d(\psi - t)}{\theta^2} = \int f(x)(q/p)^{1/2}Fdx. \tag{12c}$$

We have used the following symbols in the above equations:

$$\theta = 4q(pq)^{1/2}/(pq)', \qquad t = \int (q/p)^{1/2}dx, \tag{12d}$$

with $f(x)$ and $g(\psi - t)$ disposable functions. In particular, we can set $f(x)g(\psi - t) = h \sin^2 j(\psi - t)$ in (12b) and (12c) (h and j any constants) and obtain

$$\begin{gathered}(d\psi/dx) = (q/p)^{1/2}(1 + Fh \sin^2 (\psi - t)), \\ \psi = t - j^{-1} \cot^{-1} \int jh(q/p)^{1/2}Fdx.\end{gathered} \tag{12e}$$

Using the forms ϕ of (6) and ψ of (12) in the comparison theorem* we have the following: If in some interval $x_1 < x < x_2$ or $x_1 > x > x_2$ (over which ϕ and ψ are bounded in any given example) it can be shown that $\phi \geqq \psi$ ($\phi \leqq \psi$) near $x = x_1$, then the same inequality will continue to hold *throughout* the interval if (correspondingly)

$$\begin{gathered}(x_2 - x_1)F \sin^2 \psi \geqq (x_2 - x_1)Ff(x)g(\psi - t) \\ ((x_2 - x_1)F \sin^2 \psi \leqq (x_2 - x_1)Ff(x)g(\psi - t))\end{gathered} \tag{13}$$

throughout the interval.

For examples in which (q/p) or F lead to unbounded integrals (or to poor bounds) in forms like (7) and (8), if $c \to \alpha$ and (or) $d \to \beta$, we can take (c, d) to be interior to (α, β) and use (13) to fill in between α and c and d and β. We may thus find bounds for ϕ_c and ϕ_d in forms like (7b), or for the right-hand side of an equation such as the following (corresponding to (7b)):

$$\begin{aligned}\int_{\phi_\alpha}^{\phi_\beta} \frac{d\phi}{1 + k \sin^2 \phi} - \int_\alpha^\beta \left(\frac{q}{p}\right)^{1/2} dx \\ = \int_\alpha^\beta \left(\frac{q}{p}\right)^{1/2} \frac{(F - k) \sin^2 \phi}{1 + k \sin^2 \phi} dx.\end{aligned} \tag{14}$$

* See the second footnote on p. 166.

For example, we may bound the right-hand side of (14) (usually after preliminary use of symmetry, integration by parts, and other simple manipulations) by breaking the interval of integration into (α, c), (c, d), and (d, β), and replacing ϕ in the first and third intervals by appropriate comparison ψ's and replacing $\sin^2\phi$ in the second interval by 0 or 1 as in (7b). We should choose (c, d) so as to give the smallest (absolute value) bound for the right-hand side of (14). Using the latter method with, for example, $\psi=\theta>\phi$ near $x=-1$ in the case of Legendre's polynomials,* we can readily find such very good bounds as

$$\left(n+\frac{1}{2}-\frac{1}{\pi^2}\right)^2-\frac{1}{2}<\lambda_n^2 \quad \left(=\left(n+\frac{1}{2}\right)^2-\frac{1}{4}\right).$$

In the case mentioned before, with $p=(1-x^2)$ and $q=\lambda^2 x$, we can proceed in a very similar manner, except that we must use the Riccati equation (9) or its equivalent in order to get a suitable approximate value for ϕ at $x=0$† in terms of the value of y'/y at $x=-1$. We can quite easily get upper and lower bounds for λ_n isolating it from its neighbors λ_{n-1} and λ_{n+1}. Such isolating of the possible ranges for the parameter values is very useful if, as in this case, we can determine the values by means of a continued fraction or other transcendental equation.

Thus far we have considered the form (5) only under the special substitution $r=(pq)^{-1/2}$. We can, in general, formally simplify the bounding process by using a substitution in (5) such as

$$r=(pq)^{-1/2}\tanh k(pq)^{1/2}, \qquad k>0 \text{ any constant}, \tag{15}$$

which has the property of differing slightly from $(pq)^{-1/2}$ where $k(pq)^{1/2}$ is large, but of not introducing unboundedness in the coefficient of $(pr)^{-1}\sin^2\phi$ near end points. Under such circumstances if $\int dx/pr$ is bounded, we can use forms corresponding to (7) and (8) for obtaining the desired bounds. Using this procedure we may encounter more complicated integrals which may require series expansion or some such device in order to evaluate them. We may frequently find it profitable to use this method to bound $\phi\big|_c^d$ in (7) or (8) over an interval (c, d) in which F is bounded, instead of using comparison forms such as in (14).

In cases where the function (pq) violates the conditions required for (6) and (15), we must use other substitutions for r, being guided in

* See the second footnote on p. 168.

† See the first footnote on p. 168.

general, by the desirability of having $r \sim (pq)^{-1/2}$ in regions over which the latter varies little about a small value, and of having $(\alpha, \beta) \equiv (a, b)$ in (3). Here we should note the following general form for (5). If in (5) we set $r = \rho^2 > 0$ in $\alpha < x < \beta$, then

$$\frac{d\phi}{dx} = \frac{1}{p\rho^2} + \sin^2 \phi\rho\left(\frac{d}{dx} p \frac{d}{dx} + q - \frac{1}{p\rho^2}\right)\rho. \tag{16}$$

Further if the function $\sigma = \sigma(x)$ satisfies an equation

$$\left(\frac{d}{dx} p \frac{d}{dx} + q - \delta\right)\sigma = 0,$$

where σ and σ' are continuous and δ is piecewise continuous over $\alpha < x < \beta$, then if we define continuous functions $\rho > 0$ and ψ by the relation

$$\rho e^{i\psi} = \sigma\left(1 + i \int dx/p\sigma^2\right),$$

we have

$$\frac{d\phi}{dx} = \frac{1}{p\rho^2} + \delta\rho^2 \sin^2 \phi = \frac{d\psi}{dx} + \delta\rho^2 \sin^2 \phi \tag{17}$$

with $y = \rho(\sin \phi) \exp\{-\int \delta\rho^2 \sin \phi \cos \phi \, dx\}$.

In using (16) or (17) we ordinarily wish to choose ρ (or σ) so that the coefficient of $\sin^2 \phi$ is one-signed over (α, β), or so that bounding integrals analogous to those in (7) and (8) assume sufficiently small values.

Among substitutions other than $2s = -pr'$ (which led to (5)) we note the following:

$$r = (pq)^{-1/2}, \quad s = 0; \quad (d\phi/dx) = (q/p)^{1/2} + (\sin 2\phi)(pq)'/(4pq)^* \tag{18}$$

which is useful if, for example, $(pq)'/(4pq)$ is small compared to the continuous function $(q/p)^{1/2}$, even if the former is only piecewise continuous throughout (α, β). Special examples may suggest simple substitutions to give us the desired bounds in similar examples.

CORNELL UNIVERSITY

* A possible comparison form here is $(d\psi/dx) = (q/p)^{1/2} + f(x)g(\psi - t)$ (similar to (12b)).

ON CONTIGUOUS POINT SPACES

THEODORE HAILPERIN

In this paper we shall briefly indicate the kind of geometry obtained by a modification of one of Hausdorff's axioms for topological space. The resulting space turns out to be the contiguous point space of R. L. Moore (Rice Institute Pamphlet, vol. 23 (1936)). However, unlike Moore, we shall define *contiguity*, and we shall define it in terms of *point* and *neighborhood*. It is in terms of these two primitive indefinables that Hausdorff states his axioms for topological space (*Mengenlehre*, p. 228):

AXIOM 1. *Every point p has a neighborhood U_p. For every p, $p \in U_p$.*

AXIOM 2. *For every two neighborhoods U_p and V_p of the same point, there is a third $W_p \subset U_p \cdot V_p$.*

AXIOM 3. *Every point $q \in U_p$ has a neighborhood $U_q \subset U_p$.*

AXIOM 4. *For every p and q, $p \neq q$ implies that there exist neighborhoods U_p and U_q such that $U_p \cdot U_q = 0$.*

A contiguous point space will be defined by the Axioms 1–3 together with the following new axiom:

AXIOM 4′. *There exist points, for example, p and q, such that $p \neq q$ and such that for every U_p and U_q the common part $U_p \cdot U_q$ contains both p and q.*

This axiom is obtained by negating 4 and substituting the condition $U_p \cdot U_q \supset (p, q)$ for the weaker condition $U_p \cdot U_q \neq 0$. The property given by 4′ approximates our ordinary idea of contiguity; we set this down as a formal definition.

DEFINITION. *The point p is said to be contiguous to the point q if* (1) *$p \neq q$, and* (2) *any neighborhood of the one point contains the other.*

First, it may be pointed out, no space containing contiguous points can be a topological space. This is obvious from the method of deriving 4′. In topological space a set must have at least a denumerable infinity of points in order for it to have a limit point. This is not true for contiguous points since, if p and q are contiguous, the point p is a limit point of the set (q), which is a set containing only one point.

THEOREM 1. *No point is contiguous to itself.*

THEOREM 2. *If a point p is contiguous to q, then q is contiguous to p.*

Theorem 1 results immediately from the first part of the definition of "contiguous to" while Theorem 2 is a consequence of the symmetry of both parts.

DEFINITION. *The point p is said to be contiguous to the set M if p is contiguous to some point which is an element of M.*

THEOREM 3. *If p is contiguous to M, then p is a limit pcint of M.*

PROOF. Let $q \in M$ be a point which is contiguous to p. Then every neighborhood of p contains q, which is an element of M. Hence p is a limit point of M.

THEOREM 4. *If M is a closed point set and every point of the set H is contiguous to M, then no point of $1-M$ is a limit point of H.*

PROOF. By Theorem 3, every point of H is a limit point of M. Since M is closed, H is contained in M and consequently can have no limit points in $1-M$.

Theorems 1, 2, and 4 were taken as axioms by Moore. This was necessary since contiguity was an undefined concept in his system. By defining "contiguous to" in terms of "point" and "neighborhood" we were able to derive these three propositions from one axiom, namely, 4′. In addition we are saved the trouble of redefining a number of concepts such as "connectivity" and "boundary point"; for example, the idea of contiguity was used by Moore chiefly in relation with connectivity. Ordinarily, the sets A and B are said to be mutually separated if they satisfy the following conditions:

(1) *They are mutually exclusive.*
(2) *Neither of them contains a limit point of the other.*

To these Moore adds the further condition:

(3) *No point of A is contiguous to any point of B.*

Thus the notion of connectedness, which is defined in terms of this definition acquires a new significance in spaces containing contiguous points. A set is said to be connected if and only if it is not the sum of two mutually separated sets.

The additional condition (3) which Moore had to assume now becomes unnecessary since it results from the other two in conjunction with Theorem 3. This may be stated as a theorem:

THEOREM 5. *If M and N are connected sets and there exists a point $p \in M$ which is contiguous to N, then $M+N$ is connected.*

THEOREM 6. *If $p \in 1-M$ and p is contiguous to M, then p is a boundary point of M.*

This results immediately from Theorem 3. Here again we have derived a property which Moore had to assume (by enlarging the definition of "boundary point," loc. cit., p. 7).

THEOREM 7. *If p is contiguous to q, the set (p, q) is connected.*

THEOREM 8. *If p is contiguous to q and there exists a neighborhood containing p and q but no other contiguous points besides p and q, then the set (p, q), consisting solely of p and q, is closed, connected, and compact; that is, is a simple continuous arc from p to q.*

PROOF. The set is closed since the only limit points which (p, q) may have must be contiguous to p or q and this is ruled out by hypothesis. The set is connected by Theorem 8, and, finally, is obviously compact since it contains no infinite subset.

These theorems suffice to indicate the type of geometry which contiguity involves. For some important applications the reader is recommended to the aforementioned work of R. L. Moore.*

UNIVERSITY OF MICHIGAN

---

* At an advanced stage in his paper, Moore introduces a restrictive axiom (loc. cit. p. 39):

AXIOM D. *There do not exist 3 distinct points such that each of them is contiguous to each of the others.*

This axiom can only be proved here by assuming that our points are linearly ordered.

GENERALIZED REGULAR RINGS*

N. H. McCOY

1. **Introduction.** An element a of a ring $\mathfrak{R}$ is said to be *regular* if there exists an element x of $\mathfrak{R}$ such that $axa=a$. A ring $\mathfrak{R}$ with unit element, every element of which is regular, is a *regular ring*.† In the present note we introduce rings somewhat more general than the regular rings and prove a few results which are, for the most part, analogous to known theorems about regular rings.‡

Let $\mathfrak{R}$ denote a ring with unit element. If for every element a of $\mathfrak{R}$ there exists a positive integer n such that a^n is regular, we shall say that $\mathfrak{R}$ is *π-regular*. In general, the integer n will depend on a. If, however, there is a fixed integer m such that for all elements a of $\mathfrak{R}$, a^m is regular, we may say that $\mathfrak{R}$ is *m-regular*. In this notation, a regular ring is 1-regular.

An important example of a π-regular ring is a special primary ring, that is, a commutative ring in which every element which is not nilpotent has an inverse.§ It will be seen below that in the study of π-regular rings the special primary rings play a role similar to that of the fields in the case of regular rings.

2. **Theorems on π-regular rings.** Let $\mathfrak{R}$ be a π-regular ring, and $\mathfrak{Z}$ its center, that is, the set of all elements commutative with all elements of $\mathfrak{R}$. We now prove the first theorem:

THEOREM 1. *The center of a π-regular ring is π-regular.*

If $a \in \mathfrak{Z}$, there exists an n such that for some element x of $\mathfrak{R}$, $a^nxa^n=a^n$. Let $y=a^{2n}x^3$. Then, by a trivial modification of von Neumann's proof of the corresponding result for regular rings,‖ it follows that y is in $\mathfrak{Z}$ and that $a^nya^n=a^n$. Hence $\mathfrak{Z}$ is π-regular.

---

* Presented to the Society, September 6, 1938.

† J. von Neumann, *On regular rings*, Proceedings of the National Academy of Sciences, vol. 22 (1936), pp. 707–713.

‡ In addition to von Neumann, loc. cit., see also a paper by the present author entitled *Subrings of infinite direct sums*, Duke Mathematical Journal, vol. 4 (1938), pp. 486–494. Hereafter this paper will be referred to as S.

§ See W. Krull, *Algebraische Theorie der Ringe*, Mathematische Annalen, vol. 88 (1922), pp. 80–122; R. Hölzer, *Zur Theorie der primären Ringe*, ibid., vol. 96 (1927), pp. 719–735. A ring is *primary* if every divisor of zero is nilpotent, that is, (0) is a primary ideal.

‖ Loc. cit., p. 711.

It is a familiar result* that a ring with unit element is reducible† if and only if its center is reducible. We shall use this fact to establish the following theorem:

THEOREM 2. *A π-regular ring is irreducible if and only if its center is a special primary ring.*

In view of the remark just made, we only need to show that the commutative π-regular ring $\mathfrak{Z}$ is irreducible if and only if it is a special primary ring.

It is easy to see that a special primary ring $\mathfrak{Z}$ is irreducible. For if $\mathfrak{Z}$ is the direct sum of two proper ideals, and $1=e_1+e_2$ is the corresponding decomposition of the unit, then $e_i \neq 0$, $e_i^2=e_i$, $(i=1, 2)$, $e_1e_2=0$. Thus e_1 can be neither nilpotent nor have an inverse, in violation of the definition of a special primary ring.

Suppose now that $\mathfrak{Z}$ is an irreducible commutative π-regular ring, and that z is any element of $\mathfrak{Z}$ which is not nilpotent. We shall show that z has an inverse. For some positive integer n, there exists an x in $\mathfrak{Z}$ such that $xz^{2n}=z^n$. Now $xz^n \neq 0$, as otherwise we should have $z^n=0$. Let $e_1=xz^n$, $e_2=1-e_1$. Then it is easy to verify that $e_i^2=e_i$, $e_1e_2=0$. If $\mathfrak{Z}_i$ denotes the ideal of all elements of $\mathfrak{Z}$ of the form ce_i, $c \in \mathfrak{Z}$, $(i=1, 2)$, then $\mathfrak{Z}$ is the direct sum of the ideals $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$. Since $\mathfrak{Z}_1 \neq 0$, our assumption that $\mathfrak{Z}$ is irreducible requires that $\mathfrak{Z}_2=0$. Thus $e_2=0$, which implies that z has the inverse xz^{n-1}.

We now prove the following theorem:

THEOREM 3. *In a commutative π-regular ring $\mathfrak{R}$, every prime ideal is divisorless.*

Let $\mathfrak{p}$ be an arbitrary prime ideal in $\mathfrak{R}$. Then the ring $\mathfrak{R}/\mathfrak{p}$ contains no divisors of zero and hence is irreducible. But clearly $\mathfrak{R}/\mathfrak{p}$ is a commutative π-regular ring, and hence by the preceding theorem must be a special primary ring. However a special primary ring without divisors of zero is a field, and this implies that $\mathfrak{p}$ is divisorless.‡

The final theorem of this section now follows immediately from a theorem of Krull.§

THEOREM 4. *In a commutative π-regular ring every ideal is the intersection of its primary ideal divisors.*

---

* Cf. van der Waerden, *Moderne Algebra*, vol. 2, p. 164.

† That is, expressible as a direct sum of two proper two-sided ideals.

‡ Cf. S, Theorem 8.

§ W. Krull, *Idealtheorie in Ringen ohne Endlichkeitsbedingung*, Mathematische Annalen, vol. 101 (1929), p. 738.

3. Characterizations of commutative π-regular and m-regular rings. From the preceding theorem it follows* that a commutative π-regular ring is isomorphic to a subring of a direct sum of primary rings, there being in general an infinite number of summands. But a primary ring can be imbedded in a special primary ring,† and we thus have the theorem:

THEOREM 5. *A commutative π-regular ring is isomorphic to a subring of a direct sum of special primary rings.*

In any commutative ring, if a primary ideal $\mathfrak{q}$ has the property that whenever a finite power of an element b is in $\mathfrak{q}$, then $b^m \equiv 0$ $(\mathfrak{q})$, we shall say that $\mathfrak{q}$ is a primary ideal of *index* m. In other words, the primary ideal $\mathfrak{q}$ has index m if and only if $x^m = 0$ for every element x in the radical of $\mathfrak{R}/\mathfrak{q}$. It is obvious that a primary ideal of index m is also primary of index k, where k is any positive integer greater than m. A prime ideal is clearly a primary ideal of index 1. We may remark also that if a commutative ring is m-regular it is also $(m+1)$-regular and therefore k-regular if $k > m$. For if $a^{2m}x = a^m$, it is easily verified that

$$a^{2(m+1)}(a^{2m-1}x^2) = a^{m+1},$$

and this implies that a^{m+1} is regular.

It is now easy to establish the following generalization of a known theorem on regular rings:‡

THEOREM 6. *A necessary and sufficient condition that a commutative ring $\mathfrak{R}$, with unit element, be m-regular is that in $\mathfrak{R}$ every ideal be the intersection of its primary ideal divisors of index m.*

If $\mathfrak{R}$ is m-regular, then every primary ideal is of index m. For if $\mathfrak{q}$ is a primary ideal and $a^k \equiv 0$ $(\mathfrak{q})$, $(k > m)$, then since $a^{2m}x = a^m$, it follows that for each positive integer $i > 1$,

$$a^{im}x = a^{(i-1)m}.$$

But for some i, $a^{im} \equiv 0$ $(\mathfrak{q})$, and thus $a^{(i-1)m} \equiv 0$ $(\mathfrak{q})$. A repetition finally shows that $a^m \equiv 0$ $(\mathfrak{q})$. Hence $\mathfrak{q}$ is of index m, and Theorem 4 completes the proof of the first part of the theorem.

Conversely, suppose $\mathfrak{R}$ is a commutative ring with unit element in which every ideal is the intersection of its primary divisors of index m. Let a be an arbitrary element of $\mathfrak{R}$. We shall show that there exists

* S, Theorem 1.

† See Hölzer, loc. cit., p. 722.

‡ S, Theorem 9.

an x such that $a^{2m}x = a^m$. Let $\mathfrak{q}$ denote an arbitrary primary divisor of (a^{2m}) of index m. Then also $a^m \equiv 0$ $(\mathfrak{q})$ as follows at once from the assumption that $\mathfrak{q}$ is of index m. Hence (a^m) and (a^{2m}) have precisely the same primary ideal divisors of index m; thus, by hypothesis, it follows that $(a^m) = (a^{2m})$. That is, there exists an x such that $a^{2m}x = a^m$, and a^m is regular. Thus $\mathfrak{R}$ is m-regular.

We conclude with the following theorem:

THEOREM 7. *A necessary and sufficient condition that a commutative ring $\mathfrak{R}$, with unit element, be m-regular is that all direct indecomposable ideals be primary of index m.**

It is known† that in an arbitrary ring with unit element every ideal is the intersection of its direct indecomposable ideal divisors. If these are all primary of index m, the preceding theorem shows that $\mathfrak{R}$ is m-regular.

Suppose $\mathfrak{R}$ is m-regular, and let $\mathfrak{k}$ be a direct indecomposable ideal in $\mathfrak{R}$. Then $\mathfrak{R}/\mathfrak{k}$ is irreducible and is also m-regular. Thus, by Theorem 2, $\mathfrak{R}/\mathfrak{k}$ is a special primary ring and $\mathfrak{k}$ is therefore a primary ideal in $\mathfrak{R}$. Theorem 6 then states that $\mathfrak{k}$ is of index m, and the proof is completed.

SMITH COLLEGE

A FORMULA FOR THE COEFFICIENTS OF THE CYCLOTOMIC POLYNOMIAL‡

J. E. EATON

Despite the widespread use of the roots of unity in the solution of many mathematical questions, the problem of characterizing the irreducible equation

$$F_n(x) = x^r + a_1x^{r-1} + \cdots + a_r = 0$$

whose roots are the primitive nth roots of unity has received little attention. It is well known that $r = \phi(n)$, that $F_n(1) = p$ for $n = p^\alpha$ (where p is a prime) and $F_n(1) = 1$ otherwise. For n a power of a prime a_i is 1 or 0. In 1883 Migotti§ proved that for n a product of two primes a_i is ± 1 or 0. In 1895 Bang‖ showed that for n a product of

* Cf. S, Theorem 10.

† See S, §4.

‡ Presented to the Society, February 26, 1938.

§ Sitzungsberichte der Akademie der Wissenschaften, Vienna, (2), vol. 87 (1883), pp. 7–14.

‖ Nyt Tidsskrift for Mathematik, (B), vol. 6 (1895), pp. 6–12.

three primes the absolute value of any coefficient is less than the smallest of the primes.

I. Schur* in 1931 proved that there exist cyclotomic polynomials with coefficients arbitrarily large in absolute value. Bungers† in 1934 proved the same theorem, under the assumption that there exist infinitely many prime pairs, for n a product of three primes. In 1936 E. Lehmer‡ modified Bungers' proof so as to eliminate his unproved assumption of the existence of infinitely many prime pairs.

Although these results are extremely interesting, the only information they yield of the nature of the coefficients for an arbitrary n is that there is no bound on them independent of n. The purpose of this note is to derive a formula for the coefficients of any cyclotomic polynomial. The formula thus obtained may be of some use in numerical work. Its value in problems of a theoretical nature appears to be more limited.

Before we proceed to the actual derivation of the formula, let us first establish four lemmas.

LEMMA 1. *Consider a fixed number, say m, of indeterminates x_i. Let $\sum$ mean the usual $\sum$-function and $\sum'$ the sum as if the exponents were distinct. If*

$$a_1 = a_2 = \cdots = a_u, \quad a_{u+1} = a_{u+2} = \cdots = a_{u+v}, \cdots,$$

$$a_{k-w+1} = a_{k-w+2} = \cdots = a_k,$$

then

$$\sum x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k} = \frac{1}{u!}\frac{1}{v!}\cdots\frac{1}{w!}(-1)^k \sum_{i=1}^{k!} g_i : S$$

where g_i is an element of the symmetric group on k letters, written in cyclic form (not omitting cycles containing but one letter), and where

$$(a_1^{(1)}, a_2^{(1)}, \cdots, a_r^{(1)})(a_1^{(2)}, a_2^{(2)}, \cdots, a_s^{(2)}) \cdots (a_1^{(j)}, a_2^{(j)}, \cdots, a_t^{(j)}) : S$$
$$= (-1)^j S_{a_1^{(1)}+a_2^{(1)}+\cdots+a_r^{(1)}} S_{a_1^{(2)}+a_2^{(2)}+\cdots+a_s^{(2)}} \cdots S_{a_1^{(j)}+a_2^{(j)}+\cdots+a_t^{(j)}},$$

wherein $S_i = \sum x^i$ and $a_j^{(i)}$ is some a_k.

PROOF. We first note that $u!v!\cdots w!\sum x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r} = \sum' x_1^{a_1}x_2^{a_2}\cdots x_r^{a_r}$. The proof is by induction on k. $\sum' x_1^{a_1} = (-1)[(a_1) : S]$

* His proof is given in a paper by E. Lehmer, this Bulletin, vol. 42 (1936), p. 389. I have drawn upon this paper for all the references contained in the present article.

† Göttingen Dissertation, 1934.

‡ E. Lehmer, op. cit., pp. 389–392.

$=(-1)[-S_{a_1}]=S_{a_1}$. Hence the theorem is true for $k=1$. We assume it true for $k \leqq n$. We have

$$\sum' x_{n+1}^{a_{n+1}} \sum' x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} = \sum' x_1^{a_1+a_{n+1}} x_2^{a_2} \cdots x_n^{a_n} + \sum' x_1^{a_1} x_2^{a_2+a_{n+1}} \cdots x_n^{a_n} + \cdots + \sum' x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n+a_{n+1}} + \sum' x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} x_{n+1}^{a_{n+1}},$$

$$\begin{aligned}\sum' x_1^{a_1} x_2^{a_2} \cdots x_{n+1}^{a_{n+1}} &= \sum' x_{n+1}^{a_{n+1}} \sum' x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \\ &\quad - \sum' x_1^{a_1+a_{n+1}} x_2^{a_2} \cdots x_n^{a_n} - \cdots \\ &\quad - \sum' x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n+a_{n+1}} \\ &= -\left\{(-1)^n \sum_{i=1}^{n!} [(a_{n+1})g_i]:S + (-1)^n \sum_{i=1}^{n} h_i^{(1)}:S \right. \\ &\quad \left. + \cdots + (-1)^n \sum_{i=1}^{n!} h_i^{(n)}:S\right\},\end{aligned}$$

where $h_i^{(j)}$ is obtained from the element g_i by replacing a_j by a_j, a_{n+1}. Hence

$$\sum' x_1^{a_1} x_2^{a_2} \cdots x_{n+1}^{a_{n+1}} = (-1)^{n+1} \sum_{i=1}^{(n+1)!} g_i:S.$$

LEMMA 2. *In the symmetric group on n letters, $n>1$, the number of elements with an even number of cycles equals the number of elements with an odd number of cycles (provided cycles with but one letter are not omitted).*

PROOF. In Lemma 1, let $x_1=1$ and $x_j=0$, $(j=2, 3, \cdots, n)$. Then $S_i=1$ for all i, and

$$x_1 x_2 \cdots x_n = \frac{1}{n!}(-1)^n \sum_{i=1}^{n!} g_i:S = \frac{1}{n!}(-1)^n(J-K)$$

where J is the number of elements in the symmetric group on n letters with an even number of cycles, K the number with an odd number of cycles. Since $x_1 x_2 \cdots x_n = 0$, $J=K$.

LEMMA 3. *If n is square free and the product of k primes, then the coefficient a_1 in $F_n(x)$ is $(-1)^{k+1}$.*

PROOF. The proof is by induction on k. The theorem is obviously

true for $k=1$. We assume it to be true for $k=t$. Let m be the product of t different primes. Then $F_m(x)=x^r+(-1)^{t+1}x^{r-1}+\cdots+1$. Let p be a prime not dividing m and $\epsilon, \epsilon^2, \cdots, \epsilon^{p-1}$ the primitive pth roots of unity. Then

$$F_{mp}(x) = [x^r + (-1)^{t+1}\epsilon x^{r-1} + \cdots + \epsilon^r][x^r + (-1)^{t+1}\epsilon^2 x^{r-1} + \cdots + \epsilon^{2r}] \cdots [x^r + (-1)^{t+1}\epsilon^{p-1}x^{r-1} + \cdots + \epsilon^{(p-1)r}].$$

Expanding, we have

$$F_{mp}(x) = x^{r(p-1)} + (-1)^{t+1}(\epsilon + \epsilon^2 + \cdots + \epsilon^{p-1})x^{r(p-1)-1} + \cdots + 1.$$

Since $\epsilon+\epsilon^2+\cdots+\epsilon^{p-1}=-1$, the induction is established.

LEMMA 4. *Let $x_1, x_2, \cdots, x_r$ be the primitive nth roots of unity, where n is square free. Let $(i, n)=p_1p_2\cdots p_t=s$. Then $S_i=S_1$ if $s=1$ and $(-1)^t\phi(s)S_1$ otherwise.*

PROOF. Let $i=qs$ where $(q, n)=1$. Then

$$x_1^i + x_2^i + \cdots + x_r^i = x_1^{qs} + x_2^{qs} + \cdots + x_r^{qs} = x_{j_1}^s + x_{j_2}^s + \cdots + x_{j_r}^s.$$

But x_j^s is a primitive (n/s)th root of unity. Moreover, every (n/s)th root of unity appears in the above expression the same number of times, namely

$$\frac{\phi(n)}{\phi(n/s)} = \phi(s).$$

Hence, by Lemma 3, if k is the number of primes contained in n,

$$S_i = (-1)^{k-t+1}\phi(s) = (-1)^t\phi(s)S_1.$$

We are now in a position to investigate the main problem. We need only consider n which are square free; for let m be the largest square free number dividing n. Then $F_n(x)=F_m(x^{n/m})$.

In considering square free n, we restrict ourselves first to the case where n is the product of an odd number of primes. Then, by Lemma 3, $S_1=-1$. From Lemma 1,

$$(1) \qquad (-1)^i a_i = \sum x_1x_2\cdots x_i = \frac{1}{i!}(-1)^i \sum_{k=1}^{i!} g_k : S.$$

Suppose $g_k = c_1^{(r_1)} \cdots c_\nu^{(r_\nu)}$, where $c_\mu^{(r_\mu)}$, $(\mu=1, \cdots, \nu)$, is a cycle on r_μ letters, $r_1+r_2+\cdots+r_\nu=i$. Then $g_k : S = \prod_\mu (c_\mu^{(r_\mu)} : S) = \prod_\mu (-S_{r_\mu})$ since all exponents in (1) are 1. If r_μ is prime to n, $-S_{r_\mu} = -S_1 = 1$;

and if for all g_k, all r_μ were prime to n, we would have the relation

$$\sum_{k=1}^{i!} g_k : S = i! = G_i. \tag{2}$$

To obtain the actual value of $\sum g_k : S$, we take G_i as an "approximation" and determine corrections to be added due to certain r_μ not prime to n. A correction is first determined for cycles "associated with the length p, $p \mid n$." A cycle $c^{(r)}$ is associated with the length p if $p \mid r$, and the correction is obtained by using the "better" value $c^{(r)} : S = -(p-1)$ instead of 1, while cycles not associated with the length p are left as 1. This is done for all $p \mid n$ and $p \leqq i$. (The better value is the correct value only if $(r, n) = p$.) Then a correction is determined for cycles associated with the length pq, $pq \mid n$, $pq \leqq i$, and so on.

The first approximation (2) yields $a_i = [G_i]/i!$. In correcting for cycles associated with the length p, $p \mid n$, consider first those substitutions having a cycle of length p. They are of the form $c^{(p)}g$, where g is an element of the symmetric group on $(i-p)$ letters. Since, by Lemma 4, $S_p = (p-1)$, we have $c^{(p)}g : S = -(p-1)(g:S)$ (with the minus sign taken because of the reduction in the number of cycles operating on S). A cycle of length p may be chosen in $i!/[p(i-p)!]$ ways. If we assume that the cycles of g operating on S are evaluated for $S_j = S_1$, and if we note that we have previously evaluated, in (2), this expression as $-(-1)(g:S)$, the value of a_i becomes

$$a_i = \frac{1}{i!}\left[G_i - p\,\frac{i!}{p(i-p)!}G_{i-p}\right]. \tag{3}$$

However, if a substitution has r cycles of length p, we have counted it $C_{r,1} = r$ times.

Consider now those substitutions having a cycle of length $2p$. They are $i!/[2p(i-2p)!]$ in number and have been previously evaluated, in (2), as $-(-1)(g:S)$ instead of the better value $-(p-1)(g:S)$ (g now is an element of the symmetric group on $(i-2p)$ letters). Those substitutions having 2 cycles of length p are

$$\frac{1}{2}\,\frac{i!}{p(i-p)!}\,\frac{(i-p)!}{p(i-2p)!} = \frac{i!}{2p^2(i-2p)!}$$

in number. They have been previously evaluated, in (3), as

$$(-1)(-1)(g:S) + (2p)(-1)(g:S) = (1-2p)(g:S)$$

instead of $(p^2-2p+1)(g:S)$. If we again assume that the cycles of g

operating on S are evaluated for $S_j = S_1$, the value of a_i becomes

$$a_i = \frac{1}{i!}\left[G_i - p\,\frac{i!}{p(i-p)!}G_{i-p} - p\,\frac{i!}{2p(i-2p)!}G_{i-2p} + p^2\,\frac{i!}{2p^2(i-2p)!}G_{i-2p}\right].$$

However, if a substitution has r cycles of length $2p$, we have counted it $C_{r,1}$ times; and if a substitution has r cycles of length p, we have counted it $C_{r,2}$ times.

We proceed now to the general case, namely that of substitutions having k_1 cycles of length p, k_2 cycles of length $2p$, $\cdots$, k_s cycles of length sp. Let $\sum jk_j = k$ and $\sum k_j = t$. (We assume that we have already considered all substitutions with $(k-1)p$ or less letters involved in cycles associated with the length p.) Each combination of $(t-j)$, $(j \geqq 1)$, of the cycles of such a substitution has already introduced the correction $(-1)^{t-j}p^{t-j}$. Hence this substitution has been previously evaluated as

$$\left[(-1)^t \sum_j (-1)^j C_{t,t-j}p^{t-j}\right](g:S).$$

There are $(h/k!)i!/[p^t(i-kp)!]$ such substitutions, where h is the number of elements in the symmetric group on k letters having k_1 cycles of length 1, k_2 cycles of length 2, $\cdots$, k_s cycles of length s. If we let $k_1, k_2, \cdots, k_s$ assume all values compatible with $\sum jk_j = k$, we have

$$a_i = \frac{1}{i!}\left[G_i - \frac{i!}{(i-p)!}G_{i-p} + \left(\frac{1}{2} - \frac{1}{2}\right)\frac{i!}{(i-2p)!}G_{i-2p} + \cdots + \left(\frac{h' - h''}{k!}\right)\frac{i!}{(i-kp)!}G_{i-kp}\right],$$

where h' is the number of elements in the symmetric group on k letters having an even number of cycles and h'' the number having an odd number of cycles. However, by Lemma 2, $h' = h''$; moreover, from its definition, $G_u = u!$. Hence the correction, due to cycles associated with the length p, which must be added to the first approximation of a_i is -1. If we apply this process to cycles associated with each of the t primes less than i dividing n, the value of a_i becomes $a_i = 1 - t$.

The situation for cycles of length pq, $pq \mid n$, $pq \leqq i$, is slightly different. We have previously evaluated them as $(-p-q+1)(g:S)$. Since

their correct value by Lemma 4 is $[-(p-1)(q-1)(-1)](g:S)$, the adjustment must be made as $pq(g:S)$. An analysis similar to the case for a single prime shows that to the value of a_i previously computed must be added the following correction for cycles associated with the length pq:

$$\frac{1}{i!}\left[\frac{i!}{(i-pq)!}G_{i-pq}+\left(\frac{1}{2}+\frac{1}{2}\right)\frac{i!}{(i-2pq)!}G_{i-2pq}+\cdots\right.$$
$$\left.+\left(\frac{h'+h''}{k!}\right)\frac{i!}{(i-kpq)!}G_{i-kpq}+\cdots\right].$$

As before, h' is the number of elements in the symmetric group on k letters having an even number of cycles, and h'' the number having an odd number of cycles. Hence $h'+h''=k!$. Thus each multiple of $pq\leqq i$ adds 1 to a_i.

The proper adjustment for a cycle of length $p_1p_2\cdots p_t$, such that $p_1p_2\cdots p_t \mid n$, $p_1p_2\cdots p_t\leqq i$, is $(-1)^t p_1p_2\cdots p_t(g:S)$. This may be established quite easily by induction on t. The theorem was seen to be true for $t=1$. We assume it to be true for $t\leqq k$. Then a cycle of length $p_1p_2\cdots p_{k+1}$, $p_1p_2\cdots p_{k+1}\mid n$, $p_1p_2\cdots p_{k+1}\leqq i$, at that stage in the process when all cycles associated with a length which is a product of less than $k+1$ primes dividing n have been evaluated, has introduced the following number into a_i:

$$(4)\quad (-1)^k\left[\sum^{k+1} p_1p_2\cdots p_k-\sum^{k+1} p_1p_2\cdots p_{k-1}+\cdots+(-1)^k\right](g:S).$$

However, by Lemma 4, the correct evaluation is

$$(5)\qquad (-1)^{k+1}\phi(p_1p_2\cdots p_{k+1})(g:S).$$

But the difference between (5) and (4) is

$$(-1)^{k+1}p_1p_2\cdots p_{k+1}(g:S),$$

which then is the correction that must be made.

If t is odd, the analysis of the case for a single prime shows that each divisor $p_1p_2\cdots p_t$ of n less than or equal to i subtracts 1 from a_i. If t is even, the argument for two primes shows that each multiple of each such factor less than or equal to i adds 1 to a_i.

There remains to consider only the adjustments necessary for the simultaneous occurrence of cycles of length $r_1, r_2, \cdots, r_s$, respectively, where $r_i \mid n$. The formula for the correction, which we shall establish by induction, is $(-1)^t r_1r_2\cdots r_s(g:S)$, where t is the number

of primes (not necessarily distinct) appearing in the product $r_1r_2 \cdots r_s$. We have previously shown the formula to be true for $t=1$. We assume it to be true for $t \leqq k$. Then the s cycles of length $r_1, r_2, \cdots, r_s$, respectively, at that point when cycles associated with lengths whose product contains less than $(k+1)$ primes have all been evaluated, have introduced the following number into a_i:

$$(6)\quad (-1)^k\left[\sum^{k+1} p_1p_2 \cdots p_k - \sum^{k+1} p_1p_2 \cdots p_{k-1} + \cdots + (-1)^k\right](g:S)$$

where the summations are taken as if $p_1, p_2, \cdots, p_{k+1}$ were distinct. However, by Lemma 4, the correct evaluation is

$$(7)\qquad (-1)^{k+1}\phi(r_1)\phi(r_2) \cdots \phi(r_s)(g:S).$$

The difference between (7) and (6) is

$$(-1)^{k+1}r_1r_2 \cdots r_s(g:S),$$

and the induction is complete.

In applying the adjustment to all possible combinations of sets of cycles, where each set is associated with some r_i, we should note that the method of grouping the cycles used in analyzing the case for a single prime may still be used within each set. Hence the sum of any number, say t, of different divisors of n containing an odd number of primes plus the sum of any number of multiples of different divisors of n containing an even number of primes adds or subtracts 1 to a_i as t is even or odd. Note that $(-1)^t$ is the same whether t is defined as above or as in the previous paragraph. Also note that we must specify "different divisors," for if some $r_i=r_j$, our method of grouping would have accounted for the cycles at an earlier stage in the process.

If n is square free and the product of an even number of primes, the same general methods apply. However, certain differences arise due to the fact that $S_1=1$ instead of -1 as previously. A cycle of length p must be evaluated as $(p-1)(g:S)$. After correcting for all cycles associated with the length p, $p \mid n$, we have

$$a_i = \frac{1}{i!}\left[G_i + \frac{i!}{(i-p)!}G_{i-p} + \left(\frac{1}{2}+\frac{1}{2}\right)\frac{i!}{(i-2p)!}G_{i-2p} + \cdots + \left(\frac{h'+h''}{k!}\right)\frac{i!}{(i-kp)!}G_{i-kp} + \cdots\right].$$

Since $S_1=1$, we have $G_0=1$, $G_1=-1$, and $G_j=0$ for $j>1$. (G_u, we recall, is $\sum^{u!} g_k:S$ evaluated for $S_j=S_1$.) Hence any multiple of p equal

to i adds 1 to a_i, and any multiple of p equal to $i-1$ subtracts 1 from a_i. This is true of any product of an odd number of primes dividing n. A product of an even number of primes subtracts or adds 1 to a_i if it equals i or $i-1$, respectively. However multiples of such expressions have no effect. The extension to sums of divisors is immediate.

We have thus established the following theorem:

THEOREM. *Let* $F_n(x)=x^r+a_1x^{r-1}+\cdots+a_r=0$ *be the irreducible equation of the primitive nth roots of unity.*

A. *Let m be the largest square free number dividing n and* a_i' *a coefficient of* F_m. *Then if* $i=kn/m$, $a_i=a_k'$; *if* n/m *does not divide i,* $a_i=0$.

B. *Let n be square free and the product of an odd number of primes. Then*

$$a_i = 1 + \sum \epsilon_d$$

where the following conditions hold:

(a) *The summation is extended over all* $d \leqq i$.

(b) *The number d is the sum of any number, say t, of different divisors of n containing an odd number of primes plus the sum of any number of multiples of different divisors of n containing an even number of primes.*

(c) $\epsilon_d=(-1)^t$.

C. *Let n be square free and the product of an even number of primes. Then*

$$a_1 = -1; \qquad a_i = \sum \epsilon_d - \sum \epsilon_{d'}, \qquad i > 1,$$

where the following conditions hold:

(a) *The summations are extended over all* $d=i$ *and all* $d'=i-1$, *respectively.*

(b) *Both d and d' are the sum of any number of multiples of different divisors of n containing an odd number of primes plus the sum of any number, say t, of different divisors of n containing an even number of primes.*

(c) $\epsilon_d=(-1)^t$.

YALE UNIVERSITY

JACKSON SUMMATION OF THE FABER DEVELOPMENT*

W. E. SEWELL

1. **Introduction.** The purpose of this note is to prove the following theorem:

THEOREM. *Let C be an analytic Jordan curve in the z-plane, and let $f(z)$ be analytic in C, continuous in $\overline{C}$, the closed limited set bounded by C, and let† $f^{(p)}(z)$, $(p \geqq 0)$, satisfy a Lipschitz condition‡ of order α, $(0 < \alpha \leqq 1)$, on C. Then*

$$\left| f(z) - \sum_{\nu=0}^{n} d_{n,\nu} a_\nu P_\nu(z) \right| \leqq \frac{M}{n^{p+\alpha}}, \qquad z \text{ in } \overline{C}, \tag{1}$$

where M is a constant independent of n and z,

$$\sum_{\nu=0}^{n} a_\nu P_\nu(z)$$

is the sum of the first $n+1$ terms of the development of $f(z)$ in the Faber§ polynomials belonging to C, and $d_{n,\nu}$ is the Jackson‖ summation coefficient of order p.

In a previous paper the author¶ showed that under the above hypothesis

$$\left| f(z) - \sum_{0}^{n} a_\nu P_\nu(z) \right| \leqq M(\log n / n^{p+\alpha}), \qquad z \text{ in } \overline{C}.$$

Later John Curtiss** proved the existence of a sequence of polynomials $Q_n(z)$ of respective degrees n, $(n = 1, 2, \cdots)$, such that

$$| f(z) - Q_n(z) | \leqq M / n^{p+\alpha}.$$

---

* Presented to the Society, December 30, 1937.

† $f^{(p)}(z)$ denotes the pth derivative of $f(z)$; $f^{(0)}(z) \equiv f(z)$.

‡ $f(z)$ satisfies a Lipschitz condition of order α on C if for z_1 and z_2 arbitrary points on C we have $|f(z_1) - f(z_2)| \leqq L|z_1 - z_2|^\alpha$, where L is a constant independent of z_1 and z_2.

§ G. Faber, Mathematische Annalen, vol. 57 (1903), pp. 389–408.

‖ Dunham Jackson, Transactions of this Society, vol. 15 (1914), pp. 439–466; p. 463.

¶ This Bulletin, vol. 41 (1935), pp. 111–117; this paper will be referred to hereafter as SI.

** This Bulletin, vol. 42 (1936), pp. 873–878.

In a paper soon to appear, Walsh and the author state that for $p=0$, $(0<\alpha<1)$, the inequality $|f(z)-\sigma_n(z)| \leqq M/n^\alpha$ is valid, where $\sigma_n(z)$ is the nth arithmetic mean of the development of $f(z)$ in the Faber polynomials belonging to C. Here we extend this result by exhibiting a set of polynomials (proved by Curtiss to exist) with the prescribed degree of convergence for arbitrary p and for $0<\alpha\leqq 1$.

2. **Proof of the theorem.** Let

$$z = \psi(t) = \frac{1}{t} + b_0 + b_1 t + b_2 t^2 + \cdots = \frac{1}{t} + \mathfrak{P}(t) \tag{2}$$

map the exterior of C on the region $|t|<r$ of the complex t-plane so that the point $z=\infty$ corresponds to the point $t=0$. It follows from the analyticity of C that the right-hand side of (2) converges for $|t| \leqq r'$, $r'>r$. The Faber polynomials belonging to C are defined as follows: $P_n(z)$ is the polynomial of degree n in z such that the coefficient of z^n is unity, and, as a function of t through (2), such that the coefficients are zero for the terms in $t^{-n+1}, t^{-n+2}, \cdots, t^{-1}, t^0$; hence $P_n(z) = 1/t^n + t\mathfrak{P}_n(t)$, where $\mathfrak{P}_n(t)$ converges for $|t| \leqq r'$.

Faber (loc. cit.) and the author (SI) have shown that

$$f(z) = \sum_{\nu=0}^{\infty} a_\nu P_\nu(z), \qquad z \text{ in } \overline{C},$$

$$= \sum_{\nu=0}^{\infty} a_\nu \left(\frac{1}{t^\nu} + t\mathfrak{P}_\nu(t)\right) = \sum_{\nu=0}^{\infty} \frac{a_\nu}{t^\nu} + \sum_{1}^{\infty} c_\nu t^\nu, \qquad r \leqq |t| \leqq r'.$$

It should be noted here that (Faber, loc. cit.) $|a_\nu| \leqq M_1 r^\nu$ and $|\mathfrak{P}_\nu(t)| \leqq M_2/r'^\nu$, where M_1 and M_2 are constants.

Now we consider

$$f(z) - \sum_{\nu=0}^{n} d_{n\nu} a_\nu P_\nu(z) = f(\psi(t)) - \sum_{\nu=0}^{n} d_{n\nu} a_\nu \left(\frac{1}{t^\nu} + t\mathfrak{P}_\nu(t)\right)$$

$$= f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu} a_\nu / t^\nu$$

$$- \sum_{\nu=1}^{n} d_{n\nu} a_\nu t\mathfrak{P}_\nu(t) + \sum_{\nu=1}^{\infty} c_\nu t^\nu$$

$$= f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu} a_\nu / t^\nu$$

$$+ \sum_{\nu=1}^{n} (1 - d_{n\nu}) a_\nu t\mathfrak{P}_\nu(t) + \sum_{n+1}^{\infty} a_\nu t\mathfrak{P}_\nu(t), \qquad r \leqq |t| \leqq r'.$$

Therefore

$$\left| f(z) - \sum_{\nu=0}^{n} d_{n\nu} a_\nu \mathfrak{P}_\nu(z) \right| \leqq \left| f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu} a_\nu / t^\nu \right| + \left| \sum_{\nu=1}^{n} (1 - d_{n\nu}) a_\nu t \mathfrak{P}_\nu(t) \right| + \left| \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) \right|, \qquad r \leqq |t| \leqq r'.$$

The sum $\sum_{\nu=0}^{n} d_{n\nu} a_\nu / t^\nu$ is (SI) the Jackson summation of the first $n+1$ terms of the Taylor development of the function $f(\psi(t)) - \sum_1^\infty c_\nu t^\nu$ and consequently (Jackson, loc. cit.; Curtiss, loc. cit.)

$$\left| f(\psi(t)) - \sum_{1}^{\infty} c_\nu t^\nu - \sum_{\nu=0}^{n} d_{n\nu} a_\nu / t^\nu \right| \leqq \frac{M_3}{n^{p+\alpha}}, \qquad r \leqq |t| \leqq r'.$$

Also

$$\left| \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) \right| \leqq \sum_{n+1}^{\infty} |a_\nu| |t| |\mathfrak{P}_\nu(t)| \leqq M_1 M_2 r \sum_{n+1}^{\infty} \left(\frac{r}{r'} \right)^\nu \leqq \frac{M_4}{n^{p+\alpha}}, \qquad |t| = r,$$

since $r/r' < 1$. Furthermore*

$$|1 - d_{n\nu}| \leqq \frac{M_5 \nu^{p+1}}{n^{p+1}},$$

consequently

$$\left| \sum_{\nu=1}^{n} (1 - d_{n\nu}) a_\nu t \mathfrak{P}_\nu(t) \right| \leqq M_1 M_2 M_5 \sum_{1}^{n} \frac{\nu^{p+1}}{n^{p+1}} \left(\frac{r}{r'} \right)^\nu = \frac{M_1 M_2 M_5}{n^{p+1}} \sum_{1}^{n} \nu^{p+1} \left(\frac{r}{r'} \right)^\nu.$$

The series on the right converges, and the proof of the theorem is thus complete.

GEORGIA SCHOOL OF TECHNOLOGY

* Dunham Jackson, Transactions of this Society, vol. 15 (1914), pp. 439–466. It should be noted here that Jackson's summation coefficients vary with the derivative but not with the order of the Lipschitz condition.

A NOTE ON THE ASYMPTOTIC PROPERTIES OF ORTHOGONAL POLYNOMIALS*

R. WILSON

Let $\psi(u)$ be a function non-decreasing in the interval (0, 1) such that all the moments

$$c_k = \int_0^1 u^k d\psi(u), \qquad k = 0, 1, 2, \cdots,$$

exist, and let c_0 be positive. Then

$$f(z) \equiv \int_0^1 \frac{d\psi(u)}{z-u} = \sum_{r=0}^{\infty} \frac{c_r}{z^{r+1}} \tag{1}$$

may be developed in a continued fraction of which the denominators of the successive approximants are the Tchebichef polynomials $Q_n(z)$,† where

$$\Delta_n Q_n(z) = \begin{vmatrix} c_0 & c_1 \cdots c_n \\ c_1 & c_2 \cdots c_{n+1} \\ \cdot & \cdot \cdots \cdot \\ c_{n-1} & c_n \cdots c_{2n-1} \\ 1 & z \cdots z^n \end{vmatrix}, \qquad \Delta_n = \begin{vmatrix} c_0 & c_1 \cdots c_n \\ c_1 & c_2 \cdots c_{n+1} \\ \cdot & \cdot \cdots \cdot \\ c_{n-1} & c_n \cdots c_{2n-2} \end{vmatrix},$$

$$n = 0, 1, 2, \cdots; \Delta_0 = 1.$$

The determinants Δ_n are positive unless $\psi(u)$ has only a finite number ν of points of increase, in which case $\Delta_n = 0$ for $n > \nu$, and the continued fraction is terminating.

Shohat‡ has shown that for an extensive class of moment functions (1) we have

$$(\Delta_n/\Delta_{n+1})^{1/2} \sim 4^n, \tag{2}$$

and that for all functions of this type satisfying (2) the recurrence

---

* Presented to the Society, February 29, 1936.

† J. Shohat, *Théorie Générale des Polynomes Orthogonaux de Tchebichef*, Paris, 1934, p. 12.

‡ J. Chokhatte (Shohat), *Sur le développement de l'intégrale* $\int_a^b [p(y)/(x-y)]dy$ *en fraction continue et sur les polynomes de Tchebycheff*, Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), p. 32.

relation connecting $Q_{n+2}(z)$, $Q_{n+1}(z)$, $Q_n(z)$ leads, as $n \to \infty$, to the characteristic equation

$$\lambda^2 - \lambda\left(z - \frac{1}{z}\right) + \frac{1}{16} = 0. \tag{3}$$

From Poincaré's theorem† it follows that

$$\lim_{n\to\infty} \frac{Q_{n+1}(z)}{Q_n(z)}$$

exists and is equal to the root of larger modulus of (3) at all points z outside the range (0, 1) for which the moduli of the roots of (3) are equal. Shohat‡ has further shown that over the range (0, 1)

$$\lim_{n\to\infty} (\Delta_n/\Delta_{n+1})^{1/2} Q_n(z)/(1+\epsilon)^n = 0 \tag{4}$$

for every positive ϵ.

We prove the following theorem, which is a refinement of equation (4):

THEOREM 1. *For the type of function considered*

$$\lim_{n\to\infty} \max_{0\leq z\leq 1} |Q_n(z)|^{1/n} = \tfrac{1}{4}.$$

Following a theorem due to Perron,§ if the limiting form of the difference equation has roots of equal moduli, then although

$$\lim_{n\to\infty} \{Q_{n+1}(z)/Q_n(z)\}$$

does not exist (in general),

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n}$$

exists and is equal to the common absolute value.

At every point of (0, 1) both roots of (3) have the modulus 1/4. Hence, at every point of (0, 1)

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} = \tfrac{1}{4}. \tag{5}$$

---

* Ibid., p. 43; see also Shohat, *Théorie Générale* · · · , pp. 50–52.

† N. E. Nörlund, *Differenzenrechnung*, Berlin, 1924, pp. 300–305.

‡ See Shohat, *Sur le développement* · · · , p. 44; also *Théorie Générale* · · · , p. 52.

§ See Nörlund, p. 309.

Let $T_n(z)$ be the unique polynomial of degree n, with coefficient of z^n unity, whose maximum modulus over a given set E is a minimum. Fekete* has proved that

$$\lim_{n\to\infty} \max_{z\in E} |T_n(z)|^{1/n}$$

exists and is equal to the transfinite diameter of E.

It is well known that the transfinite diameter of the set (0, 1) is 1/4. Hence, in this case,

$$\lim_{n\to\infty} \max_{0\leq z\leq 1} |T_n(z)|^{1/n} = \tfrac{1}{4}. \tag{6}$$

From the uniqueness of $T_n(z)$ and (5) and (6), the result follows. It is of interest to note that the $Q_n(z)$, derived from $f(z)$, possess on the one hand the orthogonal properties of Tchebichef polynomials and on the other hand the properties of T-polynomials over the set of singularities of $f(z)$.

Formula (2) easily leads to the following result,† obtained by Pólya‡ in the case of a more restricted type of moment function:

THEOREM 2. *For the type of function considered*

$$\lim_{n\to\infty} \Delta^{1/n(n-1)} = \tfrac{1}{4}.$$

Theorems 1 and 2 are connected by the fact that each gives an expression§ for the transfinite diameter of the set of singularities of the moment function in terms of the coefficients of (1). The fundamental quantities $Q_n(z)$ and Δ_n there occurring are invariant under the substitutions $z' = z^{i\alpha} + c$, where α is real and c is, in general, complex.

UNIVERSITY COLLEGE OF SWANSEA,
WALES

---

* M. Fekete, *Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahlingen Koeffizienten*, Mathematische Zeitschrift, vol. 17 (1923), pp. 234–236.

† See J. Shohat, *Théorie Générale* · · · , p. 57 for an associated theorem.

‡ G. Pólya, *Über gewisse notwendige Determinantenkriterien für die Fortsetzbarkeit einer Potenzreihe*, Mathematische Annalen, vol. 99 (1928), pp. 697–700.

§ See M. Fekete, loc. cit.